

THE MULTIDIMENSIONAL TRUNCATED MOMENT PROBLEM: CARATHÉODORY NUMBERS

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ABSTRACT. Let \mathcal{A} be a finite-dimensional subspace of $C(\mathcal{X}; \mathbb{R})$, where \mathcal{X} is a locally compact Hausdorff space, and $\mathbf{A} = \{f_1, \dots, f_m\}$ a basis of \mathcal{A} . A sequence $s = (s_j)_{j=1}^m$ is called a moment sequence if $s_j = \int f_j(x) d\mu(x)$, $j = 1, \dots, m$, for some positive Radon measure μ on \mathcal{X} . Each moment sequence s has a finitely atomic representing measure μ . The smallest possible number of atoms is called the Carathéodory number $\mathcal{C}_{\mathbf{A}}(s)$. The largest number $\mathcal{C}_{\mathbf{A}}(s)$ among all moment sequences s is the Carathéodory number $\mathcal{C}_{\mathbf{A}}$. In this paper the Carathéodory numbers $\mathcal{C}_{\mathbf{A}}(s)$ and $\mathcal{C}_{\mathbf{A}}$ are studied. In the case of differentiable functions methods from differential geometry are used. The main emphasis is on real polynomials. For a large class of spaces of polynomials in one variable the number $\mathcal{C}_{\mathbf{A}}$ is determined. In the multivariate case we obtain some lower bounds and we use results on zeros of positive polynomials to derive upper bounds for the Carathéodory numbers.

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1. INTRODUCTION

The present paper continues the study of the truncated moment problem began in our previous papers [Sch15] and [dDS]. Here we investigate the Carathéodory number of moment sequences and moment cones.

Throughout this paper, we assume that \mathcal{X} is a locally compact topological Hausdorff space, \mathcal{A} is a **finite-dimensional** real linear subspace of $C(\mathcal{X}; \mathbb{R})$ and $\mathbf{A} = \{f_1, \dots, f_m\}$ is a fixed basis of the vector space \mathcal{A} .

Let $s = (s_j)_{j=1}^m$ be a real sequence and let L_s be the linear functional on \mathcal{A} defined by $L_s(f_j) = s_j, j = 1, \dots, m$. We say that s is a *moment sequence*, equivalently, L_s is a *moment functional* on \mathcal{A} , if there exists a (positive) Radon measure μ on \mathcal{X} such that f_j is μ -integrable and

$$s_j = \int f_j(x) d\mu(x) \quad \text{for } j = 1, \dots, m,$$

equivalently,

$$L_s(f) = \int_{\mathcal{X}} f(x) d\mu(x) \quad \text{for } f \in \mathcal{A}.$$

Such a measure μ is called a representing measure of s resp. L_s . The Richter–Tchakaloff Theorem (see Proposition 1 below) implies that each moment sequence has a k -atomic representing measure, where $k \leq m = \dim \mathcal{A}$. The smallest number k is called the *Carathéodory number* $\mathcal{C}_{\mathbf{A}}(s)$ and the smallest number K such that each moment sequence s has a k -atomic representing measure with $k \leq K$ is the *Carathéodory number* $\mathcal{C}_{\mathbf{A}}$.

Let L_s be a moment functional. Determining a k -atomic representing measure ν for L_s is closely related to the problem of finding quadrature or cubature formulas in

numerical integration, see for instance [DR84], [SW97]. The Carathéodory number $\mathcal{C}_A(s)$ corresponds then to the smallest possible number of nodes.

A large part of our considerations is developed in this general setup. Nevertheless we are mainly interested in the case when \mathcal{A} consists of real polynomials and \mathcal{X} is a closed subset of \mathbb{R}^n or of the projective real space $\mathbb{P}(\mathbb{R}^n)$. In this case moment sequences are usually called *truncated moment sequences* in the literature.

This paper is organized as follows. In Section 2, we define and investigate Carathéodory numbers and the cone \mathcal{S}_A of moment sequences in the case when $A \subseteq C(\mathcal{X}, \mathbb{R})$. In Section 3, we assume that the functions of A are differentiable and apply differential geometric methods to study the moment cone and Carathéodory numbers. Important technical tools are the total derivative $DS_{k,A}(C, X)$ associated with a k -atomic measure $\mu = \sum_{i=1}^k c_i \delta_{x_i}$ and the smallest number \mathcal{N}_A of atoms such that $DS_{k,A}(C, X)$ has full rank $m = |A|$. This number \mathcal{N}_A is a lower bound of the Carathéodory number \mathcal{C}_A .

The remaining four sections are concerned with polynomials. Section 4 deals with polynomials in one variable. For $A = \{1, x, \dots, x^m\}$ it is a classical fact that $\mathcal{C}_A = \lceil \frac{m}{2} \rceil$. We investigate a set A and its homogenization B with gaps, that is,

$$A = \{1, x^{d_1}, \dots, x^{d_m}\} \quad \text{and} \quad B = \{y^{2d}, x^{d_2} y^{2d-d_2}, \dots, x^{d_{m-1}} y^{2d-d_{m-1}}, x^{2d}\},$$

where $0 = d_1 < \dots < d_m = 2d$. Our main result (Theorem 45) gives sufficient conditions for the validity of the formula $\mathcal{C}_A = \mathcal{C}_B = \lceil \frac{m}{2} \rceil$.

Sections 5–7 are devoted to the multivariate case. Except from a few simple cases the Carathéodory number \mathcal{C}_A is unknown for polynomials in several variables. In Section 5 we give a new lower bound of \mathcal{C}_A and relate the number \mathcal{N}_A to the Alexander–Hirschowitz Theorem. Another group of main results of this paper is obtained in Section 6. Here we use known results on zeros of non-negative polynomials to derive upper bounds for Carathéodory numbers (Theorems 57, 59, and 62). Section 7 deals with signed Carathéodory numbers and the real Waring rank.

The multidimensional truncated moment problem was first studied in the Thesis of J. Matzke [Mat92] and independently by R. Curto and L. Fialkow [CF96a], [CF96b]. It is an active research topic, see e.g. [Ric57], [Kem68], [Rez92], [Sch15], [Lau09], [FN10], [CF13], [Fiaa], [Fiab], [dDS]. Carathéodory numbers of multivariate polynomials have been investigated by C. Riener and M. Schweighofer [RS]. Carathéodory numbers of general convex cones are studied in [Tun01].

For $r \in \mathbb{R}$ let $\lceil r \rceil$ denote the smallest integer larger or equal to r .

2. CARATHÉODORY NUMBERS: CONTINUOUS FUNCTIONS

Let δ_x be the delta measure at $x \in \mathbb{R}^n$, that is, $\delta_x(M) = 1$ if $x \in M$ and $\delta_x(M) = 0$ if $x \notin M$. By a *signed k -atomic measure* μ we mean a signed measure $\mu = \sum_{j=1}^k c_j \delta_{x_j}$, where x_1, \dots, x_k are pairwise different points of \mathbb{R}^n and c_1, \dots, c_k are nonzero real numbers. If all numbers c_1, \dots, c_k are positive, then μ is a positive measure and is called simply a *k -atomic measure*. The points x_j are called the atoms of μ . The zero measure is considered as 0-atomic measure.

A crucial result for our considerations is the *Richter–Tchakaloff Theorem* proved in [Ric57]. In the present context it can be stated as follows.

Proposition 1. *Each truncated moment sequence s of A has a k -atomic representing measure with $k \leq m = |A|$.*

Definition 2. *The moment cone $\mathcal{S}_A \equiv \mathcal{S}(A, \mathcal{X})$ is the set of all truncated \mathcal{X} -moment sequences.*

Obviously, \mathcal{S}_A is a convex cone in \mathbb{R}^m . Since the functions f_1, \dots, f_m form a vector space basis of \mathcal{A} , it follows easily that $\mathbb{R}^m = \mathcal{S}_A - \mathcal{S}_A$.

Definition 3. The Carathéodory number $\mathcal{C}_{\mathbf{A}}(s) \equiv \mathcal{C}_{\mathbf{A},\mathcal{X}}(s)$ of $s \in \mathcal{S}(\mathbf{A}, \mathcal{X})$ is the smallest k such that s has a k -atomic representing measure with all atoms in \mathcal{X} . The Carathéodory number $\mathcal{C}_{\mathbf{A}} \equiv \mathcal{C}_{\mathbf{A},\mathcal{X}}$ of the moment cone $\mathcal{S}(\mathbf{A}, \mathcal{X})$ is the smallest number $\mathcal{C}_{\mathbf{A}}$ such that each moment sequence $s \in \mathcal{S}(\mathbf{A}, \mathcal{X})$ has a k -atomic representing measure with all atoms in \mathcal{X} and $k \leq \mathcal{C}_{\mathbf{A}}$.

Definition 4. The signed Carathéodory number $\mathcal{C}_{\mathbf{A},\pm}(s) \equiv \mathcal{C}_{\mathbf{A},\mathcal{X},\pm}(s)$ of $s \in \mathbb{R}^m$ is the smallest number k such that s has a signed k -atomic representing measure with all atoms in \mathcal{X} . The signed Carathéodory number $\mathcal{C}_{\mathbf{A},\pm} \equiv \mathcal{C}_{\mathbf{A},\mathcal{X},\pm}$ is the smallest number $\mathcal{C}_{\mathbf{A},\pm}$ such that every sequence s has a signed k -atomic representing measure with all atoms in \mathcal{X} and $k \leq \mathcal{C}_{\mathbf{A},\pm}$.

Since $\mathbb{R}^m = \mathcal{S}_{\mathbf{A}} - \mathcal{S}_{\mathbf{A}}$ as noted above, Proposition 1 implies each vector $s' \in \mathbb{R}^m$ has a signed k -atomic representing measure, where $k \leq 2m$, and we have

$$(1) \quad \mathcal{C}_{\mathbf{A}}(s) \leq \mathcal{C}_{\mathbf{A}} \leq m \quad \text{for } s \in \mathcal{S}_{\mathbf{A}} \quad \text{and} \quad \mathcal{C}_{\pm}(s') \leq \mathcal{C}_{\mathbf{A},\pm} \leq 2m \quad \text{for } s' \in \mathbb{R}^m.$$

Remark 5. The above definitions of moment sequences, moment cones and Carathéodory numbers make sense for Borel functions rather than continuous functions. For instance, let x_1, \dots, x_m be pairwise different points of \mathbb{R}^n and let \mathbf{A} be the set of characteristic functions of the points x_j . Then it is easily verified that the Carathéodory number $\mathcal{C}_{\mathbf{A}}$ is equal to $m = |\mathbf{A}|$.

Definition 6. The moment curve of \mathbf{A} in \mathbb{R}^m is defined by

$$(2) \quad s_{\mathbf{A}} : \mathcal{X} \rightarrow \mathbb{R}^m, x \mapsto s_{\mathbf{A}}(x) := \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}$$

and we set

$$(3) \quad S_{k,\mathbf{A}} : (\mathbb{R}_{\geq 0})^k \times \mathcal{X}^k \rightarrow \mathbb{R}^m, (C, X) \mapsto S_{k,\mathbf{A}}(C, X) := \sum_{i=1}^k c_i \cdot s_{\mathbf{A}}(x_i),$$

where $C = (c_1, \dots, c_k)$, $X = (x_1, \dots, x_k)$.

Clearly, $s_{\mathbf{A}}(x)$ is the moment sequence of the delta measure δ_x and $S_{k,\mathbf{A}}(C, X)$ is the moment sequence with representing measure $\mu = \sum_{i=1}^k c_i \delta_{x_i}$:

$$(4) \quad S_{k,\mathbf{A}}(C, X) = \sum_{i=1}^k c_i s_{\mathbf{A}}(x_i) = \left(\int_{\mathcal{X}} f_j(x) d\mu(x) \right)_{j=1}^m.$$

By Proposition 1, each moment sequence $s \in \mathcal{S}_{\mathbf{A}}$ is of the form $S_{m,\mathbf{A}}(C, X)$ for some $(C, X) \in (\mathbb{R}_{\geq 0})^m \times \mathcal{X}^m$. Further, let us introduce a convenient notation:

$$(5) \quad \text{Pos}(\mathcal{A}, \mathcal{X}) \equiv \text{Pos}(\mathcal{X}) := \{f \in \mathcal{A} : f(x) \geq 0 \quad \text{for } x \in \mathcal{X}\}.$$

The following proposition restates a known result (see e.g. Lemma 3 and Proposition 27(i) in [dDS]).

Proposition 7. Suppose that $s \in \mathcal{S}_{\mathbf{A}}$ is a boundary point of $\mathcal{S}_{\mathbf{A}}$. Then there exists $p \in \text{Pos}(\mathcal{A}, \mathcal{K})$, $p \neq 0$, such that $L_s(p) = 0$ and each representing measure of s is supported on the set of zeros $\mathcal{Z}(p)$ of p .

The next proposition is a crucial technical ingredient of many proofs given below. The following condition is used at several places of this paper:

$$(6) \quad \text{There exists } e \in \mathcal{A} \text{ such that } e(x) \geq 0 \text{ for } x \in \mathcal{X}.$$

Proposition 8. *Let $s \in \mathcal{S}_A$ and $x \in \mathcal{X}$. Suppose that condition (6) is satisfied. Define*

$$(7) \quad c_s(x) := \sup \{c \in \mathbb{R} : (s - c \cdot s_A(x)) \in \mathcal{S}_A\}.$$

Then $c_s(x) \leq e(x)^{-1}L_s(e)$ and $(s - c_s(x)s_A(x)) \in \partial\mathcal{S}_A$.

If \mathcal{K} is compact, then the supremum in (7) is attained, the moment cone \mathcal{S}_A is closed in \mathbb{R}^m , and we have

$$(8) \quad \mathcal{C}_A \leq \max \{\mathcal{C}_A(s) : s \in \partial\mathcal{S}_A\} + 1.$$

Proof. Let $c \in \mathbb{R}$. If $(s - cs_A(x)) \in \mathcal{S}_A$, then $L_s - cl_x$ is a moment functional on \mathcal{A} and therefore $(L_s - cl_x)(e) \geq 0$, so that $c \leq e(x)^{-1}L_s(e)$. Hence $c_s(x) \leq e(x)^{-1}L_s(e)$. The definition of $c_s(x)$ implies that $s - c_s(x)s_A(x)$ belongs to the boundary of \mathcal{S}_A .

Since \mathcal{X} is compact, it was shown in [FN10] that the moment cone \mathcal{S}_A is closed in \mathbb{R}^m . We choose a sequence $(c_n)_{n \in \mathbb{N}}$ such that $s - c_n s_A(x) \in \mathcal{S}_A$ for all n and $\lim_n c_n = c_s(x)$. Then $s - c_n s_A(x) \rightarrow s - c_s(x)s_A(x)$. Since \mathcal{S}_A is closed, we have $(s - c_s(x)s_A(x)) \in \mathcal{S}_A$, that is, the supremum (7) is attained.

Note that $(s - c_s(x)s_A(x)) \in \partial\mathcal{S}_A \cap \mathcal{S}_A$. Obviously, $\mathcal{C}_A(s) \leq \mathcal{C}_A(s - c_s(x)s_A(x)) + 1$. This implies the inequality (8). \square

The following example shows that the number $c_s(x)$ is not equal to

$$(9) \quad \bar{c}_s(x) := \sup \{c \in \mathbb{R} : (s - c \cdot s_A(x)) \in \overline{\mathcal{S}_A}\}.$$

However, if $s \in \text{int } \mathcal{S}_A$, then $c_s(x) = \bar{c}_s(x)$ by Proposition 10(vi) below.

Example 9. *Set $\mathcal{X} = [-1, \pi]$,*

$$f_1(x) := 1, \quad f_2(x) := \begin{cases} 0 & x \in [-1, 0] \\ \sin x & x \in (0, \pi] \end{cases}, \quad f_3(x) := \begin{cases} x + 1 & x \in [-1, 0] \\ \cos x & x \in (0, \pi] \end{cases},$$

and $g_i = f_i|_{[-1, \pi]}$ for $i = 1, 2, 3$. Set $A = \{f_1, f_2, f_3\}$ and $B = \{g_1, g_2, g_3\}$. Then \mathcal{S}_A is closed, but \mathcal{S}_B is not closed. In fact, $\overline{\mathcal{S}_B} = \mathcal{S}_A$. Let $s = s_A(-1) = (1, 0, 0)^T$. Then $s' = s - s_A(0)/2 = (1/2, 0, -1/2)^T = s_A(\pi)/2 \in \partial\mathcal{S}_A = \partial\mathcal{S}_B$, but $s_A(\pi) \notin \mathcal{S}_A$. Thus $c_s(0) = 0$ and $\bar{c}_s(0) = 1/2$.

Recall from [Sch15] the *maximal mass function* $\rho_L(x)$ of a moment functional L :

$$(10) \quad \rho_L(x) := \sup \{\mu(\{x\}) : \mu \text{ is a representing measure of } L\}, \quad x \in \mathcal{X}.$$

Proposition 10. *Suppose that condition (6) holds and retain the notation from Proposition 8.*

- (i) $s - c \cdot s_A(x) \notin \mathcal{S}_A$ for all $c > c_s(x)$.
- (ii) If $s \in \text{int } \mathcal{S}_A$, then $s - c \cdot s_A(x) \in \text{int } \mathcal{S}_A$ for all $c < c_s(x)$.
- (iii) The map $\text{int } \mathcal{S}_A \ni s \mapsto c_s(x) \in \mathbb{R}$ is concave and continuous for all $x \in \mathcal{X}$.
- (iv) The map $\mathcal{X} \ni x \mapsto c_s(x) \in \mathbb{R}$ is continuous for all $s \in \text{int } \mathcal{S}_A$.
- (v) $c_s(x) = \rho_{L_s}(x)$.
- (vi) If $s \in \text{int } \mathcal{S}_A$, then $c_s(x) = \bar{c}_s(x)$.

Proof. (i) is clear from the definition (7).

(ii): Since s is an inner point, there exists $\varepsilon > 0$ such that $B_\varepsilon(s) \subset \text{int } \mathcal{S}_A$. From the convexity of \mathcal{S}_A it follows that

$$B_{\frac{c_s(x) - c}{c_s(x)}}(s - c \cdot s_A(x)) \subset \text{int } \mathcal{S}_A \quad \forall c < c_s(x).$$

(iii): Let $s, t \in \mathcal{S}_A$ and $\lambda \in (0, 1)$. Choose $c, c' \in \mathbb{R}$ such that $c < c_s(x)$ and $c' < c_t(x)$. Then $s - cs_A(x)$ and $t - c's_A(x)$ are in \mathcal{S}_A . Since \mathcal{S}_A is convex, we have

$$\begin{aligned} & \lambda[s - cs_A(x)] + (1 - \lambda)[t - c's_A(x)] \\ &= [\lambda s + (1 - \lambda)t] - [\lambda c + (1 - \lambda)c']s_A(x) \in \mathcal{S}_A, \end{aligned}$$

i.e., $\lambda c + (1 - \lambda)c' \leq c_{\lambda s + (1 - \lambda)t}(x)$. Taking the suprema over c and c' it follows that $\lambda c_s(x) + (1 - \lambda)c_t(x) \leq c_{\lambda s + (1 - \lambda)t}(x)$. Hence $s \mapsto c_s(x)$ is a concave function and therefore continuous on $\text{int } \mathcal{S}_A$ by [Sch14, Thm. 1.5.3].

(iv): Let $x \in \mathcal{X}$. Let K be a compact neighborhood of x and $(x_i)_{i \in I}$ a net in K such that $\lim_{i \in I} x_i = x$. Since K is compact, we have $e(y) \geq \delta > 0$ and $\|s_A(y)\| \geq \delta$ for $y \in K$. Hence $c_s(y)$ is bounded on K , say by k , by Proposition 8. Since $s_A(y)$ is continuous, there exist $M > 0$ such that $\|c_s(y)s_A(y)\| \leq M$ on K . Further, from (i) and (ii) it follows that $\partial \mathcal{S}_A \cap (s + \mathbb{R} \cdot s_A(y)) = \{s - c_s(y)s_A(y)\}$ for $y \in K$.

Define $s'_y := s - c_s(y)s_A(y)$. Then $s'_y \in B_M(s) \cap \partial \mathcal{S}_A$ for all $y \in K$. Since $\partial \mathcal{S}_A$ is closed and $B_M(s)$ is compact, $B_M(s) \cap \partial \mathcal{S}_A$ is also compact. Therefore, $(s'_{x_i})_{i \in I} \subseteq B_M(s) \cap \partial \mathcal{S}_A$ has an accumulation point, say a . Since $\partial \mathcal{S}_A$ is closed, $a \in \partial \mathcal{S}_A$. Since $c_s(x_i)$ is bounded by k and s_A is continuous,

$$|\langle v, s'_{x_i} - s \rangle| = |\langle v, -c_s(x_i)s_A(x_i) \rangle| \leq k \cdot |\langle v, s_A(x_i) \rangle| \rightarrow k \cdot |\langle v, s_A(x) \rangle| = 0$$

for all $v \perp s_A(x)$, i.e., $a - s \in [-k, k] \cdot s_A(x)$, so that $a \in s + [-k, k] \cdot s_A(x)$. Then

$$a \in \partial \mathcal{S}_A \cap (s + [-k, k] \cdot s_A(x)) \subseteq \partial \mathcal{S}_A \cap (s + \mathbb{R} \cdot s_A(x)) = \{s'_x\},$$

so $(s'_{x_i})_{i \in I}$ has a unique accumulation point s'_x . Thus $\lim_{i \in I} s'_{x_i} = s'_x$. This proves that the map $y \mapsto s'_y$ is continuous at x . Therefore,

$$\|s - s'_y\| \cdot \|s_A(y)\|^{-1} = \|c_s(y)s_A(y)\| \cdot \|s_A(y)\|^{-1} = \|c_s(y)\| = c_s(y)$$

is continuous at x . Since $x \in \mathcal{X}$ was arbitrary, $x \mapsto c_s(x)$ is continuous on \mathcal{X} .

(v): Let $c \in \mathbb{R}$ be such that $\tilde{s} := s - c \cdot s_A(x) \in \mathcal{S}_A$. Then $L_s = L_{\tilde{s}} + c \cdot \delta_x$. Hence there is a representing measure μ of s such that $c \leq \mu(\{x\}) \leq \rho_{L_s}(x)$. Taking the supremum over c yields $c_s(x) \leq \rho_{L_s}(x)$.

Assume that $c_s(x) < \rho_{L_s}(x)$. By the definition of $\rho_{L_s}(x)$, there exist a $c \in (c_s(x), \rho_{L_s}(x))$ and a representing measure μ of s such that $\mu(\{x\}) = c$. Then $\tilde{\mu} := \mu - c \cdot \delta_x$ is a positive Radon measure representing $\tilde{s} = s - c \cdot s_A(x)$. But $\tilde{s} \notin \mathcal{S}_A$ by (i), a contradiction. This proves that $c_s(x) \not< \rho_{L_s}(x)$. Thus, $c_s(x) = \rho_{L_s}(x)$.

(vi): Since $s \in \text{int } \mathcal{S}_A$, it follows from (i) and (ii) that

$$\partial \mathcal{S}_A \cap (s + \mathbb{R} \cdot s_A(x)) = \{s'_x = s - c_s(x)s_A(x)\}.$$

Both numbers $s - c_s(x)s_A(x)$ and $s - \bar{c}_s(x)s_A(x)$ belong to the set on left hand side set. Hence they are equal and therefore $c_s(x) = \bar{c}_s(x)$. \square

From Proposition 10(iii) we easily derive that the supremum in (10) is attained if \mathcal{X} is compact. This was proved in [Sch15, Prop. 6] by using the weak topology on the set of representing measures and the Portmanteau Theorem.

The following example shows that (iv) is false in general if $s \in \partial \mathcal{S}_A$.

Example 11. Let $\{x_1, \dots, x_{10}\}$ be the zero set of the Robinson polynomial, \mathcal{A} the homogeneous polynomials of degree 6 on $\mathbb{P}(\mathbb{R}^2)$, and $s := \sum_{i=1}^{10} s_A(x_i)$. By Theorem 18 and Example 18 in [dDS], s is determinate. Therefore,

$$\rho_s(x) = c_s(x) = \begin{cases} 1 & \text{for } x \in \{x_1, \dots, x_{10}\}, \\ 0 & \text{else.} \end{cases}$$

If K is not compact, then the supremum in (7) is not attained in general. This is shown by the following simple example.

Example 12. Let $\mathcal{X} = \mathbb{R}$, $\mathcal{A} = \{1, x, x^2\}$. Set $s = (1, 0, 1)^T = \frac{1}{2}(s_A(-1) + s_A(1))$. Clearly, $s_A(0) = (1, 0, 0)^T$. Then $c_s(0) = 1$, but $s' = s - c_s(0)s_A(0) = (0, 0, 1)^T$ is not in \mathcal{S}_A .

The following theorem improves the first equality in (1) and Proposition 1.

Theorem 13. *Suppose that condition (6) holds. If $m \geq 2$ and \mathcal{X} has at most $m - 1$ path-connected components, then $\mathcal{C}_A \leq m - 1$.*

Proof. Obviously, the Carathéodory number \mathcal{C}_A depends only on the linear span A , but not on the particular basis A of $\mathcal{A} = \text{Lin } A$. Hence we can assume without loss of generality that $e = f_m$. Since $e(x) > 0$ on \mathcal{X} by assumption, $b_j := f_j e^{-1} \in C(\mathcal{X})$ for $j = 1, \dots, m$. Set $B = \{b_1, \dots, b_m\}$.

Let s be a moment sequence of B . First we prove that s has a finitely atomic representing measure of at most $m - 1$ atoms. Upon normalization we can assume that $s_m = 1$. By Proposition 1, s has a k -atomic measure $\mu = \sum_{j=1}^k c_j \delta_{x_j}$, where $k \leq m$ and $x_j \in \mathcal{X}$ and $c_j > 0$ for all j . If $k < m$, we are done, so we can assume that $k = m$. Since \mathcal{X} consists of at most $m - 1$ path-connected components, it follows that at least two points x_i , say x_1 and x_2 , are in the same component, say \mathcal{X}_1 , of \mathcal{X} . Then there is a connecting path $\gamma : [0, 1] \rightarrow \mathcal{X}_1$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. For $t \in [0, 1]$ we denote by Δ_t the simplex in $\mathbb{R}^{m-1} \times \{1\}$ spanned by the points $s_B(x_1), s_B(\gamma(t)), s_B(x_3), \dots, s_B(x_m)$. Since $s_m = 1$, we have $\sum_{j=1}^m c_j = 1$. Hence $s := (s_1, \dots, s_m)$ belongs to the convex hull of $s_B(x_1), s_B(x_2), s_B(x_3), \dots, s_B(x_m)$, that is, s is in the simplex Δ_1 . By decreasing t to 0 it follows from the continuity of b_i that there exists a $t_0 \in [0, 1]$ such that s belongs to the boundary of the simplex Δ_{t_0} . Then s is a convex combination of at most $m - 1$ vertices. This yields a k -representing measure $\tilde{\mu}$ of s with $k \leq m - 1$.

Now we show that each moment sequence of A has a k -atomic representing measure with $k \leq m - 1$. This in turn implies the assertion $\mathcal{C}_A \leq m - 1$. Let s' be a moment sequence of A and let μ' be a finitely atomic representing measure of s' . Let s be the moment sequence of B given by the measure $e(x)d\mu$. As shown in the preceding paragraph, s has a k -atomic representing measure ν , where $k \leq m - 1$. Then $e(x)^{-1}d\nu$ is a k -atomic representing measure of s' . \square

Corollary 14. *Let $\mathcal{A} = \{p \in \mathbb{R}[x_1, \dots, x_n] : \deg(p) \leq d\}$ and $\mathcal{X} = \mathbb{R}^n$. Then*

$$\mathcal{C}_A \leq |A| - 1 = \binom{n+d}{n} - 1.$$

We give two somewhat pathological examples. Example 15 shows that the assertion of Theorem 13 is not true if the assumption on the function $e(x)$ is omitted.

Example 15. *Set*

$$\varphi(x) := \begin{cases} x & \text{for } x \in [0, 1], \\ -x + 2 & \text{for } x \in (1, 2], \\ 0 & \text{elsewhere.} \end{cases}$$

$\varphi_1(x) := \varphi(x)$, $\varphi_2(x) := \varphi(x - 1)$, $\varphi_3(x) := \varphi(x - 2)$. Then $A := \{\varphi_1, \varphi_2, \varphi_3\} \subset C(\mathbb{R})$. Using the moment sequence $s = (1, 1, 1)$ we find that $\mathcal{C}_A = 3$.

Example 16 gives a three-dimensional moment cone with $\mathcal{C}_A = 1$. A slight modification of this idea yields for $m \in \mathbb{N}$ an m -dimensional space \mathcal{A} such that $\mathcal{C}_A = 1$.

Example 16. *Let x_L and y_L be the coordinate functions of a space filling curve [Sag94, Ch. 5], i.e., $x_L, y_L : [0, 1] \rightarrow [0, 1]$ are continuous, nowhere differentiable on the Cantor set \mathcal{C} , differentiable on $[0, 1] \setminus \mathcal{C}$, and the curve*

$$(x_L, y_L) : [0, 1] \rightarrow [0, 1]^2$$

is surjective. Set $A := \{x_L, y_L, 1\}$ and $\mathcal{X} = [0, 1]$. Then

$$(*) \quad s_A([0, 1]) = [0, 1]^2 \times \{1\}$$

and the moment cone $\mathcal{S}_A = \{(x, y, z) : z \geq 0, 0 \leq x \leq z, 0 \leq y \leq z\}$ is full-dimensional. Clearly, () implies that $\mathcal{C}_A = 1$.*

Remark 17. In this paper the vector space \mathcal{A} is finite-dimensional. However the definitions of the moment cone and the Carathéodory number can be extended to infinite-dimensional vector spaces \mathcal{A} . The following example shows that even in this case it is possible that $\mathcal{C}_A = 1$. Let $A = \{\varphi_n\}_{n \in \mathbb{N}}$ be the coordinate functions of the \aleph_0 -dimensional Schönberg space filling curve [Sag94, Ch. 7], i.e., φ_n is continuous and nowhere differentiable on $[0, 1]$ for all n , and set $\varphi_0 = 1$. Then

$$(*) \quad (\varphi_n)_{n \in \mathbb{N}_0} : [0, 1] \rightarrow \{1\} \times [0, 1]^{\mathbb{N}}$$

is surjective. The moment cone $\mathcal{S}_A = \{(x_n)_{n \in \mathbb{N}_0} : 0 \leq x_n \leq x_0\}$ is full dimensional, closed, and $\mathcal{C}_A = 1$ from (*).

Theorem 18. Let $p \in \mathcal{A}$ and $x_1, \dots, x_k \in \mathcal{X}, k \in \mathbb{N}$. Suppose that $p(x) \geq 0$ for $x \in \mathcal{X}$, $\mathcal{Z}(p) = \{x_1, \dots, x_k\}$ and the set $\{s_A(x_i) : i = 1, \dots, k\}$ is linearly independent. Then $\mathcal{C}_A \geq k$.

Proof. Let $s = \sum_{i=1}^k s_A(x_i)$. Clearly, $L_s(p) = 0$ and hence $\text{supp } \mu \subseteq \mathcal{Z}(p) = \{x_1, \dots, x_k\}$ for any representing measure μ of s by Proposition 7. Assume there is an at most $(k-1)$ -atomic representing measure μ . Without loss of generality we assume that $x_1 \notin \text{supp } \mu$, so μ is of the form $\mu = \sum_{i=2}^k c_i \delta_{x_i}$, $c_i \geq 0$. Then

$$0 = s - s = \sum_{i=1}^k s_A(x_i) - \sum_{i=2}^k c_i s_A(x_i) \Rightarrow s_A(x_1) = \sum_{i=2}^k (c_i - 1) s_A(x_i).$$

Since the set $\{s_A(x_i) : i = 1, \dots, k\}$ is linear independent, this is a contradiction. Therefore, $k = \mathcal{C}_A(s) \leq \mathcal{C}_A$. \square

Applications of the previous theorem will be given in Examples 31 and 63. Deeper results on the connections between the Carathéodory number and the zeros of positive polynomials are treated in Section 6.

We derive some useful facts which will be used several times. We investigate some properties of the set

$$\mathcal{S}_k := \text{range } \mathcal{S}_{k,A}$$

of moment sequences which are given by measures of at most k atoms.

Lemma 19. For fixed $k \in \mathbb{N}$ the following are equivalent:

- (i) \mathcal{S}_k is convex, or equivalently, $\mathcal{S}_k + \mathcal{S}_k \subseteq \mathcal{S}_k$.
- (ii) $\mathcal{S}_k = \mathcal{S}_{k+1}$.
- (iii) $k \geq \mathcal{C}_A$.

Proof. (i) \Rightarrow (ii): Let $s = (1 - \lambda)s_1 + \lambda s_A(x) \in \mathcal{S}_{k+1}$ with $s_1, s_A(x) \in \mathcal{S}_k$. Since \mathcal{S}_k is convex, $s \in \mathcal{S}_k$. Hence $\mathcal{S}_k = \mathcal{S}_{k+1}$.

(ii) \Rightarrow (iii): Let $s = s_0 + \lambda_1 s_A(x_1) + \dots + \lambda_l s_A(x_l) \in \mathcal{S}_{k+l}$ be an arbitrary moment sequence. Set $s_i := s_0 + \lambda_1 s_A(x_1) + \dots + \lambda_i s_A(x_i)$. Then

$$\begin{aligned} s_1 = s_0 + \lambda_1 s_A(x_1) &\in \mathcal{S}_{k+1} = \mathcal{S}_k \Rightarrow s_2 = s_1 + \lambda_2 s_A(x_2) \in \mathcal{S}_{k+1} = \mathcal{S}_k \\ &\vdots \\ \Rightarrow s = s_{l-1} + \lambda_l s_A(x_l) &\mathcal{S}_{k+1} = \mathcal{S}_k \end{aligned}$$

Thus $\mathcal{C}_A \leq k$.

(iii) \Rightarrow (ii): Since $\mathcal{C}_A \leq k$, we have $\mathcal{S}_{\mathcal{C}_A} \subseteq \mathcal{S}_k \subseteq \mathcal{S}_{\mathcal{C}_A}$. Here the last inclusion follows from the minimality of \mathcal{C}_A . Hence, $\mathcal{S}_k = \mathcal{S}_{\mathcal{C}_A}$ is convex. \square

An immediate consequence of the preceding lemma are the following inclusions:

$$(11) \quad \{0\} = \mathcal{S}_0 \subsetneq \mathcal{S}_1 \subsetneq \dots \subsetneq \mathcal{S}_{\mathcal{C}_A} = \mathcal{S}_{\mathcal{C}_A+j}, \quad j \in \mathbb{N}.$$

Proposition 20. (i) $\mathcal{C}_A = \min\{k : \mathcal{S}_k \text{ is convex}\} = \min\{k : \mathcal{S}_k = \mathcal{S}_{k+1}\}$.

(ii) For each $k = 0, 1, \dots, \mathcal{C}_A$ there is a moment sequence s such that $\mathcal{C}_A(s) = k$.

Proof. (i) follows at once from the minimality of \mathcal{C}_A in Lemma 19.

(ii): By (11), we have $\mathcal{S}_{k-1} \subsetneq \mathcal{S}_k$ for $k = 0, \dots, \mathcal{C}_A$, where we set $\mathcal{S}_{-1} := \emptyset$. Therefore, $\mathcal{S}_k \setminus \mathcal{S}_{k-1} \neq \emptyset$. \square

Proposition 21. *Suppose that condition (6) is satisfied.*

(i) *The cone \mathcal{S} is pointed, that is, $\mathcal{S} \cap (-\mathcal{S}) = \{0\}$.*

(ii) *If \mathcal{S}_1 is closed, then \mathcal{S}_k is closed for all k .*

(iii) *If the set \mathcal{X} is compact, then \mathcal{S}_k is closed for all k .*

Proof. (i): Suppose that $s, -s \in \mathcal{S}$. Using that $e(x) > 0$ on \mathcal{X} we conclude that $L_s(e) \geq 0$ and $L_{-s}(e) = -L_s(e) \geq 0$, so $L_s(e) = 0$ and therefore $s = 0$.

(ii): The proof follows by induction. Assume \mathcal{S}_1 and \mathcal{S}_k is closed for some k . We show that \mathcal{S}_{k+1} is also closed.

Let $(s_n)_{n \in \mathbb{N}}$ be a sequence of \mathcal{S}_{k+1} such that $s_n \rightarrow s \in \overline{\mathcal{S}_{k+1}}$. We can write $s_n = \alpha_n x_n + \beta_n y_n$ such that $x_n \in \mathcal{S}_k, y_n \in \mathcal{S}_1$, $\alpha_n, \beta_n \in [0, +\infty)$, and $\|x_n\| = \|y_n\| = 1$ for all n . Since \mathcal{S}_k and \mathcal{S}_1 are closed, the sets $\mathcal{S}_k \cap B_1(0)$ and $\mathcal{S}_1 \cap B_1(0)$ are both compact. Hence we can find a subsequence (n_i) such that $x_{n_i} \rightarrow x \in \mathcal{S}_k \cap B_1(0)$, and $y_{n_i} \rightarrow y \in \mathcal{S}_1 \cap B_1(0)$. Let us assume for a moment that the sequences (α_{n_i}) and (β_{n_i}) are bounded. There is a subsequence n_{i_j} such that $\alpha_{n_{i_j}} \rightarrow \alpha \in [0, +\infty)$ and $\beta_{n_{i_j}} \rightarrow \beta \in [0, +\infty)$. Then $s_{n_{i_j}} \rightarrow s = \alpha x + \beta y \in \mathcal{S}_{k+1}$. Thus, \mathcal{S}_{k+1} is closed.

We show that the sequence (β_{n_i}) is unbounded if (α_{n_i}) is unbounded. Taking the standard scalar product $\langle \cdot, \cdot \rangle$ in \mathbb{R}^m , we can uniquely write $y_n = y_n^\perp + y_n^\parallel$ with $x_n \parallel y_n^\parallel$, $x_n \perp y_n^\perp$. Then

$$\|s_{n_i}\|^2 = \|\alpha_{n_i} x_{n_i} + \beta_{n_i} y_{n_i}^\parallel\|^2 = \|\alpha_{n_i} x_{n_i} + \beta_{n_i} y_{n_i}^\parallel\|^2 + \beta_{n_i}^2 \|y_{n_i}^\perp\|^2 \geq \beta_{n_i}^2 \|y_{n_i}^\perp\|^2.$$

Since (s_{n_i}) converges, the sequence $(\|s_{n_i}\|)$ is bounded by some k . Thus,

$$k \geq \|\alpha_{n_i} x_{n_i} + \beta_{n_i} y_{n_i}^\parallel\| \geq |\alpha_{n_i} \|x_{n_i}\| - \beta_{n_i} \|y_{n_i}^\parallel\|| = |\alpha_{n_i} - \beta_{n_i} \|y_{n_i}^\parallel\||$$

and if (α_{n_i}) is unbounded, so (β_{n_i}) is unbounded. The same reasoning shows that (α_{n_i}) is unbounded if (β_{n_i}) is unbounded.

If the sequence (β_{n_i}) is unbounded, $(y_{n_i}^\perp)$ converges to 0 and hence $y = -x$. Since \mathcal{S} is pointed by (i), this implies $x = y = 0$, a contradiction to $\|x\| = \|y\| = 1$. This completes the proof.

(iii): By (ii) it suffices to prove that \mathcal{S}_1 is closed. Clearly, condition (6) implies that $s_A(x) \neq 0$ for all $x \in \mathcal{X}$. Since $A \subseteq C(\mathcal{X}, \mathbb{R})$ and \mathcal{X} is compact, we have $\|s_A\|^{-1} s_A \in C(\mathcal{X}, S^{m-1})$ (S^{m-1} denotes the unit sphere in \mathbb{R}^m) and $\text{range } \|s_A\|^{-1} s_A$ is closed. Hence, $\mathcal{S}_1 \equiv \mathbb{R}_{\geq 0} \cdot \text{range } \|s_A\|^{-1} s_A$ is closed. \square

More on the moment cone can be found in Proposition 30.

3. CARATHEODORY NUMBERS: DIFFERENTIABLE FUNCTIONS

In the rest of this paper we assume that $\mathcal{X} = \mathbb{R}^n$ or $\mathbb{P}(\mathbb{R}^n)$ and \mathcal{A} is a finite-dimensional linear subspace of $C^r(\mathbb{R}^n; \mathbb{R})$, $r \in \mathbb{N}$.

Clearly, $S_{k,A}$ in Definition 6 is a C^r -map of $\mathbb{R}_{\geq 0}^k \times \mathbb{R}^{kn}$ into \mathbb{R}^m . Let $DS_{k,A}$ denote its total derivative. We can write

$$\begin{aligned} (12) \quad DS_{k,A} &= (\partial_{c_1} S_{k,A}, \partial_{x_1^{(1)}} S_{k,A}, \dots, \partial_{x_1^{(n)}} S_{k,A}, \partial_{c_2} S_{k,A}, \dots, \partial_{x_k^{(n)}} S_{k,A}) \\ &= (s_A(x_1), c_1 \partial_1 s_A|_{x=x_1}, \dots, c_1 \partial_n s_A|_{x=x_1}, s_A(x_2), \dots, c_k \partial_n s_A|_{x=x_k}). \end{aligned}$$

The following number is crucial in what follows.

Definition 22.

$$(13) \quad \mathcal{N}_A := \min\{k \in \mathbb{N} : \text{rank } DS_{k,A} = m\},$$

i.e., \mathcal{N}_A is the smallest number k of atoms such that $DS_{k,A}$ has full rank $m = |A|$.

A lower bound for \mathcal{N}_A is given by the following proposition.

Proposition 23. *We have $\left\lceil \frac{|A|}{n+1} \right\rceil \leq \mathcal{N}_A$. If all functions f_i are homogeneous of the same degree, then $\left\lceil \frac{|A|}{n} \right\rceil \leq \mathcal{N}_A$.*

Proof. Since $DS_{k,A}$ has $|A|$ rows and each atom contributes $n+1$ columns, we need at least $k \geq \frac{|A|}{n+1}$ atoms for full rank. Thus, $\mathcal{N}_A \geq \left\lceil \frac{|A|}{n+1} \right\rceil$.

If all functions f_i are homogeneous of degree r , then $f_i(\lambda x) = \lambda^r f_i(x)$ and so $\delta_{\lambda x} = \lambda^r \delta_x$. Hence $DS_{1,A}$ has rank at most d and kernel dimension at least 1. Therefore, at least $k \geq \frac{|A|}{n}$ atoms are needed, so that $\mathcal{N}_A \geq \left\lceil \frac{|A|}{n} \right\rceil$. \square

Example 24. Let $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi \neq 0$, and $\text{supp}(\varphi) \subseteq (0, 1)$. Set $\varphi_i(x) := \varphi(x - i + 1)$ for $i = 1, \dots, m \in \mathbb{N}$ and $A := \{\varphi_1, \dots, \varphi_m\}$. Then $\partial s_A(x) = \varphi'_i(x)e_i$ for $x \in (i-1, i)$ or 0 otherwise. Then $\mathcal{N}_A = \mathcal{C}_A = m$.

Theorem 25. Suppose that $A \subseteq C^1(\mathbb{R}^n, \mathbb{R})$. Then

$$(14) \quad \mathcal{C}_{A,\pm} \leq 2\mathcal{N}_A.$$

Set $C = (1, \dots, 1) \in \mathbb{R}^{\mathcal{N}_A}$. There exists $X \in \mathbb{R}^{\mathcal{N}_A \cdot n}$ and an open neighborhood U of (C, X) such that for every $\varepsilon > 0$ there are $(C_\varepsilon, X_\varepsilon) \in U$ and $\lambda_\varepsilon \in \mathbb{R}$ such that

$$(15) \quad s = \lambda_\varepsilon (S_{A,\mathcal{N}_A}(C, X) - S_{A,\mathcal{N}_A}(C_\varepsilon, X_\varepsilon)).$$

Proof. It clearly suffices to prove the second part of the theorem. The first assertion follows then from the second.

Since $DS_{\mathcal{N}_A}$ has full rank, there is a $(C, X) \in \mathbb{R}_{>0}^{\mathcal{N}_A} \times \mathbb{R}^{\mathcal{N}_A \cdot n}$ such that $DS_{\mathcal{N}_A}(C, X)$ has full rank. Since scaling the columns of $DS_{\mathcal{N}_A}(C, X)$ does not change the rank, we can assume without loss of generality that $C = (1, \dots, 1)$. Since the determinant is continuous there is an open neighborhood U of (C, X) such that

$$(*) \quad S_{\mathcal{N}_A,A}(C, X) \in \text{int } S_{\mathcal{N}_A,A}(U).$$

Let $s \in \mathbb{R}^m$. By $(*)$ there is a $(C_\varepsilon, X_\varepsilon) \in U$ such that $S_{\mathcal{N}_A,A}(C, X) - S_{\mathcal{N}_A,A}(C_\varepsilon, X_\varepsilon)$ is a multiple of s , i.e., (15) holds for some $\lambda_\varepsilon \in \mathbb{R}$. \square

Definition 26. Let $n, k \in \mathbb{N}$ with $k \geq \mathcal{N}_A$. A k -atomic measure (C, X) on \mathbb{R}^n is called *regular* (for $S_{k,A}$) iff $DS_{k,A}(C, X)$ has full rank. Otherwise the measure (C, X) is called *singular* (for $S_{k,A}$).

A real sequence $s = (s_\alpha)_{\alpha \in A}$ is called *regular* iff $S_{k,A}^{-1}(s)$ is empty (that is, s is not a moment sequence) or consists solely of regular measures. Otherwise, s is called *singular*.

Theorem 27. Suppose that $A \subset C^r(\mathbb{R}^n; \mathbb{R})$ and $r > \mathcal{N}_A \cdot (n+1) - m$. Then

$$(16) \quad \mathcal{N}_A \leq \mathcal{C}_A.$$

Further, the set of moment sequences s which can be represented by less than \mathcal{N}_A atoms has $|A|$ -dimensional Lebesgue measure zero in \mathbb{R}^m .

Proof. By Proposition 23 we have $r > \mathcal{N}_A \cdot (n+1) - m \geq 0$, so that $r \geq 1$.

The moment sequences which can be represented by less than \mathcal{N}_A atoms are singular values. Hence the second assertion follows from Sard's Theorem [Sar42].

To prove (16) assume to the contrary that $\mathcal{C}_A < \mathcal{N}_A$. Then every moment sequence in the moment cone is singular. This is a contradiction to Sard's Theorem since the moment cone has non-empty interior. \square

Remark 28. Theorem 27 also holds for the signed Carathéodory number with *verbatim* the same proof. With Theorem 25 we get

$$(17) \quad \mathcal{N}_A \leq \mathcal{C}_{A,\pm} \leq 2\mathcal{N}_A$$

for $A \subset C^r(\mathbb{R}^n; \mathbb{R})$ and $r > \mathcal{N}_A \cdot (n+1) - m$. Without these conditions the lower bound needs not to hold, neither for \mathcal{C}_A nor for $\mathcal{C}_{A,\pm}$, see [Fed69, pp. 317–318].

Proposition 29. Suppose that $A \subseteq C^1(\mathbb{R}^n, \mathbb{R})$ and $\{x \in \mathbb{R}^n : s_A(x) = 0\}$ is bounded. Let $\gamma \in C^1(\mathbb{R}^n, \mathbb{R})$ be such that $\gamma(x) \geq 1$ and $\lim_{|x| \rightarrow \infty} \frac{f_i(x)}{\gamma(x)} = 0$ for $i = 1, \dots, m$ and let s be a moment sequence of A . Set

$$(18) \quad \Gamma_{l,c}(s) := S_{A, \mathcal{C}_A(s)+l}^{-1}(s) \cap \left\{ (C, X) \in \mathbb{R}_{\geq 0}^{\mathcal{C}_A(s)+l} \times \mathbb{R}^{n \cdot (\mathcal{C}_A(s)+l)} \mid \sum_i c_i \gamma(x_i) \leq c \right\}$$

with $l \in \mathbb{N}_0$ and $c \geq 0$. Then:

- (i) $\Gamma_{l,c}(s)$ is closed for all $l \in \mathbb{N}_0$ and $c \geq 0$.
- (ii) $\Gamma_{0,c}(s)$ is compact for all $c \geq 0$.

If, in addition, s is regular, then:

- (iii) $\exists c \geq 0 : \Gamma_{l,c}(s) \text{ unbounded} \Leftrightarrow l \geq 1$.
- (iv) $\Gamma_{l,c}(s) \text{ compact } \forall c \geq 0 \Leftrightarrow l = 0$.

Proof. (i): If $f \in C(\mathbb{R}^n, \mathbb{R}^m)$ and $K \subseteq \mathbb{R}^m$ is closed, then $f^{-1}(K)$ is closed by the continuity of f . Since both intersecting sets in (18) are of the form $f^{-1}(K)$, they are closed and so is their intersection.

(ii): Suppose $\Gamma_{0,c}$ is non-empty. Since $\Gamma_{0,c}$ is closed by (i), it suffices to show that it is bounded. Assume to the contrary that it is unbounded and let $(C^{(i)}, X^{(i)})$ be a sequence such that $\lim_{i \rightarrow \infty} \|(C^{(i)}, X^{(i)})\| = \infty$. Since

$$0 \leq c_j^{(i)} \leq c_j^{(i)} \gamma(x_j) \leq \sum_l c_l^{(i)} \gamma(x_l) \leq c$$

the sequence $(C^{(i)})$ is bounded. The sequence $(X^{(i)})$ is unbounded. After renumbering and passing to subsequences we can assume that $c_j^{(i)} \rightarrow c_j^*$ for all j , $x_j^{(i)} \rightarrow x_j^*$ for $j = 1, \dots, k$ and $\|x_j^{(i)}\| \rightarrow \infty$ for $j = k+1, \dots, \mathcal{C}_A(s)$ as $i \rightarrow \infty$. Since

$$s = S_{A, \mathcal{C}_A}((C^{(i)}, X^{(i)})) = \sum_{j=1}^k c_j^{(i)} s_A(x_j^{(i)}) + \sum_{i=k+1}^{\mathcal{C}_A} c_j^{(i)} s_A(x_j^{(i)})$$

for all i , it follows that

$$\begin{aligned} s &= \lim_{i \rightarrow \infty} \sum_{j=1}^k c_j^{(i)} s_A(x_j^{(i)}) + \lim_{i \rightarrow \infty} \sum_{j=k+1}^{\mathcal{C}_A} c_j^{(i)} s_A(x_j^{(i)}) \\ &= \sum_{j=1}^k c_j^* s_A(x_j^*) + \lim_{i \rightarrow \infty} \underbrace{\sum_{j=k+1}^{\mathcal{C}_A} \frac{s_A(x_j^{(i)})}{\gamma(x_j^{(i)})}}_{\rightarrow 0} \cdot \underbrace{c_j^{(i)} \gamma(x_j^{(i)})}_{\leq c} \\ &= \sum_{j=1}^k c_j^* s_A(x_j^*). \end{aligned}$$

Therefore, $\mu^* = ((c_1^*, \dots, c_k^*), (x_1^*, \dots, x_k^*))$ is a k -atomic representing measure of s with $k \leq \mathcal{C}_A(s)$. By the minimality of $\mathcal{C}_A(s)$, $k = \mathcal{C}_A(s)$. Hence all sequence $(x_j^{(i)})$ are bounded which is a contradiction. Thus $\Gamma_{0,c}(s)$ is bounded.

It is clear that (iii) and (iv) are equivalent. Thus it suffices to prove (iii).

(iii) “ \Rightarrow ”: By (ii), if $\Gamma_{l,c}(s)$ is unbounded, we find a k -atomic representing measure with $k < \mathcal{C}_A(s) + l$, i.e., $l \geq 1$.

(iii) “ \Leftarrow ”: We will show that there is a $c > 0$ such that for every $x \in \mathbb{R}^n$ there is a representing measure μ in $\Gamma_{1,c}(s)$ which has x as an atom. This will prove that $\Gamma_{1,c}$, hence also $\Gamma_{l,c}$, is unbounded for $l \geq 1$.

Let $\mu_0 = (C_0, X_0) = ((c_{0,1}, \dots, c_{0,\mathcal{C}_A(s)}), (x_{0,1}, \dots, x_{0,\mathcal{C}_A(s)}))$ be a representing measure of s . Set

$$c := \int \gamma \, d\mu_0 + 1.$$

Since s is regular, all representing measures have full rank. Hence there exist variables y_1, \dots, y_m from $c_1, \dots, c_{\mathcal{C}_A(s)}, x_{1,1}, \dots, x_{\mathcal{C}_A(s),n}$ such that $D_y S(\mu_0)$ is a square matrix with full rank. Then

$$F((C, X), t) = S_{\mathcal{C}_A(s), A}((C, X)) - s + t \cdot s_A(x)$$

is a C^1 -function such that $F((C_0, X_0), 0) = 0$ and $D_y F((C_0, X_0)) = D_y S(\mu_0)$ is bijective. Thus, F fulfills all assumptions of the implicit function theorem, hence there are an $\varepsilon > 0$ and a C^1 -function $(C(t), X(t))$ such that $F((C(t), X(t)), t) = 0$ for all $t \in (-\varepsilon, \varepsilon)$. Since $c_{i,0} > 0$, there is $t_0 \in (0, \varepsilon)$ such that $c_i(t_0) > 0$ for all i , so

$$\mu(t_0) = \sum_{i=1}^{\mathcal{C}_A(s)} c_i(t_0) \delta_{x_i(t_0)} + t_0 \delta_x \quad \text{with} \quad \int \gamma \, d\mu(t_0) \leq c$$

is a $(\mathcal{C}_A(s) + 1)$ -atomic representing measure of s which has x as an atom. \square

(iii) and (iv) no longer hold if s is singular. E.g., let s be moment sequence of the measure $\mu = \sum_{i=1}^{10} \delta_{x_i}$ where x_i are the ten zeros of the Robinson polynomial, then $S_{k,A}^{-1}(s) \subseteq [0, 10]^k \times \{x_1, \dots, x_{10}\}^k$ is compact for all $k \geq 10$.

The next proposition summarizes a number of basic properties of the sets \mathcal{S}_k and the Carathéodory number. Recall that $B_\rho(t)$ is the ball with center t and radius ρ in \mathbb{R}^m .

Proposition 30. (i) Suppose that \mathcal{S}_{k-1} is closed for some k , $\mathcal{N}_A \leq k \leq \mathcal{C}_A$.

Then there exist a moment sequence s and an $\varepsilon > 0$ such that $\mathcal{C}_A(t) = k$ for all $t \in B_\varepsilon(s)$.

(ii) $s \in \text{int } \mathcal{S}_{\mathcal{C}_A}$ if and only if there exists (C, X) such that $S_A(C, X) = s$ and $\text{rank } DS_A(C, X) = |A|$.

(iii) $s \in \partial \mathcal{S}_{\mathcal{C}_A}$ if and only if $\text{rank } DS_A(C, X) < |A|$ for all (C, X) such that $S_A(C, X) = s$.

(iv) Suppose that $\mathcal{N}_A < \mathcal{C}_A$ and \mathcal{S}_k is closed for all $k = \mathcal{N}_A, \dots, \mathcal{C}_A$. Then for each such k there exists $s \in \text{int } \mathcal{S}_{\mathcal{C}_A}$ such that all k -atomic representing measures of s are singular, but s has a regular representing measure with at least $k+1$ atoms.

(v) Suppose that $\mathcal{N}_A < \mathcal{C}_A$, \mathcal{S}_k is closed for $k = \mathcal{C}_A - 1, \mathcal{C}_A$ and $\mathcal{S}_{\mathcal{C}_A} \neq \mathbb{R}^{|A|}$. If $\mathbb{R}^{|A|} \setminus \mathcal{S}_{\mathcal{C}_A-1}$ is path-connected, there exists $s \in \partial \mathcal{S}_{\mathcal{C}_A}$ such that $\mathcal{C}_A(s) = \mathcal{C}_A$.

Proof. (i): Fix such a number k and assume $(\text{int } \mathcal{S}_k) \setminus \mathcal{S}_{k-1} = \emptyset$. Then we have $\mathcal{S}_{k-1} \supseteq \text{int } \mathcal{S}_k \supseteq \text{int } \mathcal{S}_{k-1}$. Taking the closure, $\mathcal{S}_{k-1} = \overline{\mathcal{S}_{k-1}} \supseteq \overline{\mathcal{S}_k} \supseteq \overline{\mathcal{S}_{k-1}} = \mathcal{S}_{k-1}$, so that $\mathcal{S}_k \subseteq \mathcal{S}_{k-1}$ which contradicts (11). Thus, $(\text{int } \mathcal{S}_k) \setminus \mathcal{S}_{k-1} \neq \emptyset$.

(ii): “ \Leftarrow ”: Let (C, X) be a full rank measure of s . Then a neighborhood U of (C, X) is mapped onto a neighborhood of s , that is, s is an inner point.

“ \Rightarrow ”: Let s be an inner point. Choose ν such that $S_A(\nu)$ has full rank. Since s is an inner point, there exists $\varepsilon > 0$ such that $s' := s - \varepsilon \cdot S_A(\nu)$ is also an inner point. In particular, s' is a moment sequence. Let μ' be a representing measure of s' . Then $\mu = \mu' + \varepsilon \cdot \nu$ is a representing measure of s and has full rank, since already $DS_A(\nu)$ has full rank.

(iii) follows from (ii).

(iv): Let $s \in \text{int } \mathcal{S}_k \subseteq \text{int } \mathcal{S}_{C_A}$. By (i), there exists $t \in (\text{int } \mathcal{S}_{C_A}) \setminus \mathcal{S}_k$ for all $k = \mathcal{N}_A, \dots, C_A - 1$. Then $[s, t] := \{\lambda s + (1 - \lambda)t \mid \lambda \in [0, 1]\} \subseteq \text{int } \mathcal{S}_{C_A}$ by the convexity of \mathcal{S}_{C_A} . Therefore, since $s \in \text{int } \mathcal{S}_k$ but $s \notin \text{int } \mathcal{S}_k$, we have

$$\text{int } \mathcal{S}_{C_A} \cap \partial \mathcal{S}_k \supseteq [s, t] \cap \partial \mathcal{S}_k \neq \emptyset.$$

Hence there exists $s \in \text{int } \mathcal{S}_{C_A} \cap \partial \mathcal{S}_k$. Then all k -atomic representing measures of s are singular. Otherwise, a full rank k -atomic measures implies that s is an inner point of \mathcal{S}_k . But, by (iv), s has a regular l -atomic measure with $l > k$.

(v): Let $s \in \text{int } \mathcal{S}_{C_A} \setminus \mathcal{S}_{C_A-1}$ by (i) and $t \in \mathbb{R}^{|\mathcal{A}|} \setminus \mathcal{S}_{C_A-1}$. Since $s, t \in \mathbb{R}^{|\mathcal{A}|} \setminus \mathcal{S}_{C_A-1}$, they are path-connected, so there exists a continuous path $\gamma : [0, 1] \rightarrow \mathbb{R}^{|\mathcal{A}|} \setminus \mathcal{S}_{C_A-1}$ with $\gamma(0) = s$ and $\gamma(1) = t$. But since $s = \gamma(0) \in \text{int } \mathcal{S}_{C_A}$, $t = \gamma(1) \notin \text{int } \mathcal{S}_{C_A}$, and $\gamma([0, 1]) \subseteq \mathbb{R}^{|\mathcal{A}|} \setminus \mathcal{S}_{C_A-1}$, we have $\gamma([0, 1]) \cap (\partial \mathcal{S}_{C_A} \setminus \mathcal{S}_{C_A-1}) \neq \emptyset$. Therefore, $\partial \mathcal{S}_{C_A} \setminus \mathcal{S}_{C_A-1} \neq \emptyset$. \square

In Sections 4 and 6 we derive upper bounds of \mathcal{C}_A by using Proposition 8 and the inequality (8). As (v) implies, this inequality can be strict, since the Carathéodory number \mathcal{C}_A can be attained at a boundary point, see the following example.

Example 31. *The (homogeneous) Motzkin polynomial*

$$M(x, y, z) = z^6 + x^4 y^2 + x^2 y^4 - 3x^2 y^2 z^2$$

has the 6 projective roots

$$\mathcal{Z}(M) = \{(1, 1, 1), (1, 1, -1), (1, -1, 1), (1, -1, -1), (1, 0, 0), (0, 1, 0)\}.$$

We consider the truncated moment problem on the projective space $\mathbb{P}(\mathbb{R}^2)$ for

$$\mathbf{B} := \{z^6 + x^4 y^2 + x^2 y^4 - 3x^2 y^2 z^2, x^6, y^6, z^6, x^5 y, x^5 z, x^4 y z\}.$$

Clearly, $M \in \text{lin } \mathbf{B}$. Since M is non-negative and has a discrete set of roots, $s = \sum_{\xi \in \mathcal{Z}(M)} s_{\mathbf{B}}(\xi)$ is a boundary point of the closed moment cone. The matrix

$$(s_{\mathbf{B}}(\xi))_{\xi \in \mathcal{Z}(M)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix}$$

has rank 6, i.e., the set $\{s_{\mathbf{B}}(\xi)\}_{\xi \in \mathcal{Z}(M)}$ is linearly independent. Hence $\mathcal{C}_{\mathbf{B}}(s) = 6$ by Theorem 18 and $\mathcal{C}_{\mathbf{B}} \leq 6 = |\mathbf{B}| - 1$ by Theorem 13. Thus, $\mathcal{C}_{\mathbf{B}} = \mathcal{C}_{\mathbf{B}}(s) = 6$, that is, the Carathéodory number is attained at the boundary moment sequence s .

Next we derive an upper bound for the Carathéodory number in terms of zeros of positive elements of \mathcal{A} . For the rest of this section we assume that \mathcal{X} is a closed subset of \mathbb{R}^n or $\mathbb{P}(\mathbb{R}^n)$ and $\mathbf{A} \subseteq C^1(\mathcal{X}, \mathbb{R})$. By the latter we mean that there exists an open subset \mathcal{U} of \mathbb{R}^n or $\mathbb{P}(\mathbb{R}^n)$ such that $\mathcal{X} \subseteq \mathcal{U}$ and $\mathbf{A} \subseteq C^1(\mathcal{U}, \mathbb{R})$.

Definition 32. Let $\mathcal{M}_{\mathbf{A}}$ be the largest number k obeying the following property:

$(*)_k$: There exist $f \in \mathcal{A}$ and $x_1, \dots, x_k \in \mathcal{Z}(f)$ such that $f(x) \geq 0$ on \mathcal{X} and $\{s_{\mathbf{A}}(x_i)\}_{i=1, \dots, k}$ is linearly independent ($DS_{k, \mathbf{A}}((1, \dots, 1), (x_1, \dots, x_k))$ does not have full rank).

From the definition it is clear that $\mathcal{M}_{\mathbf{A}}$ is the largest dimension an exposed face of $\mathcal{S}_{\mathbf{A}}$.

Proposition 33. For each $s \in \partial \mathcal{S}_{\mathbf{A}} \cap \mathcal{S}_{\mathbf{A}}$ we have $\mathcal{C}_{\mathbf{A}}(s) \leq \mathcal{M}_{\mathbf{A}}$.

Proof. In this proof we abbreviate $N := \mathcal{C}_A(s)$. Let $\mu = \sum_{i=1}^N c_i \delta_{x_i}$ be an N -atomic representing measure of s . Since $s \in \partial \mathcal{S}_A \cap \mathcal{S}_A$, there exists $f \in \mathcal{A}$, $f \neq 0$, such that $f(x) \geq 0$ on \mathcal{X} and $L_s(f) = 0$. From the latter it follows that $\text{supp } \mu \subseteq \mathcal{Z}(f)$ and hence $x_1, \dots, x_N \in \mathcal{Z}(f)$. Further, by Proposition 30(iii), $s \in \partial \mathcal{S}_A \cap \mathcal{S}_A$ implies that $DS_{N,A}(C, X)$ does not have full rank $|A|$. Since $c_i > 0$ for all i , we have $\text{rank } DS_{N,A}(C, X) = \text{rank } DS_{N,A}((1, \dots, 1), X)$. Finally, by Theorem 18 the set $\{s_A(x_i)\}_{i=1, \dots, N}$ is linearly independent. Thus, property $(*)_N$ in Definition 32 holds, so that $\mathcal{C}_A(s) = N \leq \mathcal{M}_A$. \square

Theorem 34. *Suppose that \mathcal{X} is a compact subset of \mathbb{R}^n or $\mathbb{P}(\mathbb{R}^n)$, condition (6) is satisfied, and $A \subseteq C^1(\mathcal{X}, \mathbb{R})$. Then*

$$\mathcal{C}_A \leq \mathcal{M}_A + 1.$$

Proof. The assumptions of this theorem ensure that Proposition 8 applies. Hence the assertion follows by combining Proposition 33 with the inequality (8). \square

4. CARATHÉODORY NUMBERS: ONE-DIMENSIONAL MONOMIAL CASE

For the one-dimensional truncated moment problem the number \mathcal{N}_A can be calculated from the formula for the Vandermonde determinant.

Lemma 35. *Let $A := \{1, x, \dots, x^n\}$, where $n \in \mathbb{N}$.*

(i) *If $n = 2k - 1$, $k \in \mathbb{N}$, then*

$$(19) \quad \det DS_{k,A} = c_1 \cdots c_k \cdot \prod_{1 \leq i < j \leq k} (x_j - x_i)^4.$$

(ii) *If $n = 2k$, $k \in \mathbb{N}$, then*

$$(20) \quad \det(DS_{k-1,A}, s_A(x_k)) = c_1 \cdots c_{k-1} \cdot \prod_{1 \leq i < j \leq k-1} (x_j - x_i)^4 \cdot \prod_{i=1}^{k-1} (x_k - x_i)^2.$$

(iii) $\mathcal{N}_A = \lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n+1}{2} \rceil$.

Proof. We carry out the proofs in the odd case $n = 2k - 1$. The even case $n = 2k$ is derived in a similar manner.

We have $\partial_{c_i} S_k(C, X) = s_A(x_i)$ and $\partial_{x_i} S_k(C, X) = c_i s'_A(x_i)$. Therefore,

$$(21) \quad \det DS_{k,A} = c_1 \cdots c_k \det(s_A(x_1), s'_A(x_1), \dots, s_A(x_k), s'_A(x_k))$$

and we compute

$$\begin{aligned} & \det(s_A(x_1), s'_A(x_1), \dots, s_A(x_k), s'_A(x_k)) \\ &= \lim_{h_1 \rightarrow 0} \dots \lim_{h_k \rightarrow 0} \det \left(s_A(x_1), \frac{s_A(x_1 + h_1) - s_A(x_1)}{h_1}, \dots, \frac{s_A(x_k + h_k) - s_A(x_k)}{h_k} \right) \\ &= \lim_{h_1 \rightarrow 0} \dots \lim_{h_k \rightarrow 0} \frac{\det(s_A(x_1), s_A(x_1 + h_1), \dots, s_A(x_k + h_k))}{h_1 \cdots h_k} \\ &= \lim_{h_1 \rightarrow 0} \dots \lim_{h_k \rightarrow 0} \frac{\prod_{i=1}^k \left(h_i \prod_{j=i+1}^k (x_j + h_j - x_i)(x_j - x_i)(x_j - x_i - h_i)(x_j + h_j - x_i - h_i) \right)}{h_1 \cdots h_k} \\ &= \prod_{1 \leq i < j \leq k} (x_j - x_i)^4. \end{aligned}$$

Combined with (21), this yields (19).

We choose the numbers x_i pairwise different and all c_i positive. Then the determinants in (19) and (20) are non-zero. Hence $\det DS_{k,A} \neq 0$ and therefore $\mathcal{N}_A = k = \lfloor \frac{n}{2} \rfloor + 1$. \square

Example 36. *H. Richter [Ric57] has shown that for the one-dimensional truncated moment problem $\mathbf{A} = \{1, x, \dots, x^d\}$ the Carathéodory number is $C_{\mathbf{A}} = \lceil \frac{d+1}{2} \rceil$. This result will also follow from Theorem 45 below. If we take this equality for granted and combine it with Lemma 35(iii), then we obtain*

$$\mathcal{N}_{\mathbf{A}} = \left\lfloor \frac{d}{2} \right\rfloor + 1 = \left\lceil \frac{d+1}{2} \right\rceil = C_{\mathbf{A}}.$$

Now we turn to the general case and assume that

$$(22) \quad \mathbf{A} = \{x^{d_1}, \dots, x^{d_m}\}, \quad \text{where } 0 \leq d_1 < d_2 < \dots < d_m, \quad d_1, \dots, d_m \in \mathbb{N}_0.$$

Then we compute

$$\begin{aligned} f_{\mathbf{A}}(x_1, \dots, x_m) &:= \det(s_{\mathbf{A}}(x_1), \dots, s_{\mathbf{A}}(x_m)) = |(x_i^{d_j})_{i,j=1,\dots,m}| \\ &= \begin{vmatrix} x_1^{d_1} & x_2^{d_1} & \dots & x_m^{d_1} \\ x_1^{d_2} & x_2^{d_2} & \dots & x_m^{d_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{d_m} & x_2^{d_m} & \dots & x_m^{d_m} \end{vmatrix} = (x_1 \cdots x_m)^{d_1} \cdot \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1^{d_2-d_1} & x_2^{d_2-d_1} & \dots & x_m^{d_2-d_1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{d_m-d_1} & x_2^{d_m-d_1} & \dots & x_m^{d_m-d_1} \end{vmatrix}. \end{aligned}$$

From the latter equation it follows that each linear polynomial $x_j - x_i$, $j \neq i$, divides the polynomial $f_{\mathbf{A}}$. Hence there exists a polynomial $p_{\mathbf{A}}$ such that

$$(23) \quad \begin{aligned} f_{\mathbf{A}}(x_1, \dots, x_m) &= |(x_i^{d_j})_{i,j=1,\dots,m}| \\ &= (x_1 \cdots x_m)^{d_1} \prod_{1 \leq i < j \leq m} (x_j - x_i) \cdot p_{\mathbf{A}}(x_1, \dots, x_m) \end{aligned}$$

The polynomial $p_{\mathbf{A}}$ is uniquely determined by (23). It is homogeneous with degree

$$(24) \quad \deg p_{\mathbf{A}} = d_1 + \dots + d_m - md_1 - \frac{m(m-1)}{2}.$$

Such polynomials $p_{\mathbf{A}}$ are called Schur polynomials. They are well studied in the literature, see e.g. [Mac95]. For these Schur polynomials it is known that

$$(25) \quad p_{\mathbf{A}}(x_1, \dots, x_m) = \sum_{\alpha} x^{\alpha},$$

where α ranges over some Young tableaux. In particular, (25) implies that all non-zero coefficients of $p_{\mathbf{A}}$ are positive.

Example 37. (1) $\mathbf{A} = \{x, x^4, x^7\}$. Then we compute

$$\det(s_{\mathbf{A}}(x_1), s_{\mathbf{A}}(x_2), s_{\mathbf{A}}(x_3)) = x_1 x_2 x_3 \prod_{1 \leq i < j \leq 3} (x_j - x_i) \cdot p_{\mathbf{A}}(x_1, x_2, x_3),$$

where

$$p_{\mathbf{A}}(x_1, x_2, x_3) = (x_1^2 + x_1 x_2 + x_2^2)(x_1^2 + x_1 x_3 + x_3^2)(x_2^2 + x_2 x_3 + x_3^2).$$

(2) $\mathbf{A} = \{x, x^2, x^6\}$. Then

$$\det(s_{\mathbf{A}}(x_1), s_{\mathbf{A}}(x_2), s_{\mathbf{A}}(x_3)) = x_1 x_2 x_3 \prod_{1 \leq i < j \leq 3} (x_j - x_i) \cdot p_{\mathbf{A}}(x_1, x_2, x_3),$$

where

$$\begin{aligned} p_{\mathbf{A}}(x_1, x_2, x_3) &= x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2 + x_2^3 \\ &\quad + x_2^2 x_3 + x_2 x_3^2 + x_3^3. \end{aligned}$$

(3) $\mathbf{A} = \{1, x, x^2, x^6\}$. Then

$$\det(s_{\mathbf{A}}(x_1), s_{\mathbf{A}}(x_2), s_{\mathbf{A}}(x_3), s_{\mathbf{A}}(x_4)) = \prod_{1 \leq i < j \leq 3} (x_j - x_i) \cdot p_{\mathbf{A}}(x_1, x_2, x_3, x_4)$$

with

$$\begin{aligned} p_{\mathbf{A}}(x_1, x_2, x_3, x_4) = & x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1^2 x_4 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_2 x_4 \\ & + x_1 x_3^2 + x_1 x_3 x_4 + x_1 x_4^2 + x_2^3 + x_2^2 x_3 + x_2^2 x_4 + x_2 x_3^2 \\ & + x_2 x_3 x_4 + x_2 x_4^2 + x_3^3 + x_3^2 x_4 + x_3 x_4^2 + x_4^3. \end{aligned}$$

(4) $\mathbf{A} = \{1, x^2, x^3, x^5, x^6\}$. Then

$$\det((s_{\mathbf{A}}(x_i))_{i=1}^5) = \prod_{1 \leq i < j \leq 5} (x_j - x_i) \cdot p_{\mathbf{A}}(x_1, x_2, x_3, x_4, x_5)$$

with

$$p_{\mathbf{A}}(x_1, x_2, x_3, x_4) = \sum_{\alpha \in \Omega} x^\alpha + 3 \sum_{\alpha \in \Phi} x^\alpha,$$

$$\Omega = \{\text{all permutations of } (2, 2, 1, 1, 0)\}, \Phi = \{\text{all permutations of } (2, 1, 1, 1, 1)\}.$$

Definition 38. Assume that \mathbf{A} is as in (22) and $p_{\mathbf{A}}$ is defined by (23). Set

$$q_{\mathbf{A}}(x_1, \dots, x_k) := p_{\mathbf{A}}(x_1, x_1, \dots, x_k, x_k)$$

if $m = 2k$ is even and

$$q_{\mathbf{A},i}(x_1, \dots, x_k) := p_{\mathbf{A}}(x_1, x_1, \dots, x_{i-1}, x_{i-1}, x_i, x_{i+1}, x_{i+1}, \dots, x_k, x_k)$$

for all $i = 1, \dots, k$ if $m = 2k - 1$ is odd.

Lemma 39. (i) If m is even then $q_{\mathbf{A}}$ is symmetric.

(ii) If m is odd then $q_{\mathbf{A},i}(x_1, \dots, x_k) = q_{\mathbf{A},k}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k, x_i)$ for all $i = 1, \dots, k$.

Proof. (i): Since the Schur polynomial $p_{\mathbf{A}}$ is symmetric, so is $q_{\mathbf{A}}$.

(ii): We derive

$$\begin{aligned} q_{\mathbf{A},i}(x_1, \dots, x_k) &= p_{\mathbf{A}}(x_1, x_1, \dots, x_{i-1}, x_{i-1}, x_i, x_{i+1}, x_{i+1}, \dots, x_k, x_k) \\ &= p_{\mathbf{A}}(x_1, x_1, \dots, x_{i-1}, x_{i-1}, x_{i+1}, x_{i+1}, \dots, x_k, x_k, x_i) \\ &= q_{\mathbf{A},k}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k, x_i). \end{aligned} \quad \square$$

In the odd case it suffices to prove formula (27) below. All other determinants are then obtained by interchanging variables and Lemma 39(ii).

Lemma 40. Suppose that \mathbf{A} is of the form (22).

(i) If $m = 2k$ is even, then

$$(26) \quad \det DS_{k,\mathbf{A}}(c_1, \dots, c_k, x_1, \dots, x_k) = c_1 \cdots c_k \cdot (x_1 \cdots x_k)^{2d_1} \prod_{1 \leq i < j \leq k} (x_j - x_i)^4 \cdot q_{\mathbf{A}}(x_1, \dots, x_k).$$

(ii) If $m = 2k - 1$ is odd, then

$$(27) \quad \det(DS_{k-1,\mathbf{A}}, s_{\mathbf{A}}(x_k)) = c_1 \cdots c_{k-1} \cdot (x_1 \cdots x_{k-1})^{2d_1} x_k^{d_1} \cdot \prod_{1 \leq i < j \leq k-1} (x_j - x_i)^4 \cdot \prod_{i=1}^{k-1} (x_k - x_i)^2 \cdot q_{\mathbf{A},k}(x_1, \dots, x_k).$$

(iii) $\mathcal{N}_{\mathbf{A}} = \lceil \frac{m}{2} \rceil$.

Proof. (i): Clearly, $\partial_{c_i} S_k(C, X) = s_A(x_i)$ and $\partial_{x_i} S_k(C, X) = c_i s'_A(x_i)$. By the linearity of the determinant, the factor $c_1 \cdots c_k$ can be taken out, so we can assume without loss of generality that $c_1 = \dots = c_k = 1$.

Let $m = 2k$. We proceed in a similar manner as in the proof of Lemma 35. Using (23) we derive

$$\begin{aligned}
& \det(s_A(x_1), s'_A(x_1), \dots, s'_A(x_k)) \\
&= \lim_{h_1 \rightarrow 0} \dots \lim_{h_k \rightarrow 0} \det \left(s_A(x_1), \frac{s_A(x_1 + h_1) - s_A(x_1)}{h_1}, \dots, \frac{s_A(x_k + h_k) - s_A(x_k)}{h_k} \right) \\
&= \lim_{h_1 \rightarrow 0} \dots \lim_{h_k \rightarrow 0} \frac{\det(s_A(x_1), s_A(x_1 + h_1), \dots, s_A(x_k + h_k))}{h_1 \cdots h_k} \\
&= \lim_{h_1 \rightarrow 0} \dots \lim_{h_k \rightarrow 0} \frac{f_A(x_1, x_1 + h_1, x_2, x_2 + h_2, \dots, x_k + h_k)}{h_1 \cdots h_k} \\
&= \lim_{h_1 \rightarrow 0} \dots \lim_{h_k \rightarrow 0} \frac{\prod_{i=1}^k \left(h_i \prod_{j=i+1}^k (x_j + h_j - x_i)(x_j - x_i)(x_j - x_i - h_i)(x_j + h_j - x_i - h_i) \right)}{h_1 \cdots h_k} \\
&\quad \times (x_1(x_1 + h_1) \cdots x_k(x_k + h_k))^{d_1} p_A(x_1, x_1 + h_1, x_2, x_2 + h_2, \dots, x_k, x_k + h_k) \\
&= (x_1 \cdots x_k)^{2d_1} \cdot \prod_{i < j} (x_j - x_i)^4 \cdot p_A(x_1, x_1, x_2, x_2, \dots, x_k) \\
&= (x_1 \cdots x_k)^{2d_1} \cdot \prod_{i < j} (x_j - x_i)^4 \cdot q_A(x_1, \dots, x_k).
\end{aligned}$$

(ii): The proof in the odd case $n = 2k - 1$ is similar.

(iii): Since q_A is not the zero polynomial and all nonzero coefficients are positive, there are x_1, \dots, x_k such that $\det(DS_{k,A})(x_1, \dots, x_k) \neq 0$. Then $DS_{k,A}$ has full rank, so that $\mathcal{N}_A = k = \lceil m/2 \rceil$. \square

Now we turn to the homogeneous case and set

$$(28) \quad \mathbf{B} = \{x^{d_1} y^{d_m - d_1}, \dots, x^{d_m}\}, \quad \text{where } 0 \leq d_1 < d_2 < \dots < d_m, \quad d_i \in \mathbb{N}_0.$$

Example 41. (1) In the case $\mathbf{B} = \{xy^7, x^4y^4, x^7y\}$ we have

$$\det((s_{\mathbf{B}}(x_i, y_i))_{i=1}^3) = x_1 y_1 x_2 y_2 x_3 y_3 \cdot \prod_{1 \leq i < j \leq 3} (x_j y_i - x_i y_j) \cdot p_{\mathbf{B}}(x_1, y_1, x_2, y_2, x_3, y_3)$$

with

$$p_{\mathbf{B}}(x_1, y_1, x_2, y_2, x_3, y_3) = \prod_{1 \leq i < j \leq 3} (x_i^2 y_j^2 + x_i y_i x_j y_j + x_j^2 y_i^2).$$

(2) $\mathbf{B} = \{xy^5, x^2y^4, x^6\}$. Then we have

$$\det((s_{\mathbf{B}}(x_i, y_i))_{i=1}^3) = x_1 x_2 x_3 \cdot \prod_{1 \leq i < j \leq 3} (x_j y_i - x_i y_j) \cdot p_{\mathbf{B}}(x_1, y_1, x_2, y_2, x_3, y_3)$$

with

$$\begin{aligned}
p_{\mathbf{B}}(x_1, x_2, x_3) &= x_1^3 y_2^3 y_3^3 + x_1^2 y_1 x_2 y_2^2 y_3^3 + x_1^2 y_1 y_2^3 x_3 y_3^2 + x_1 y_1^2 x_2^2 y_2 y_3^3 \\
&\quad + x_1 y_1^2 x_2 y_2^2 x_3 y_3^2 + x_1 y_1^2 y_2^3 x_3^2 y_3 + y_1^3 x_2^3 y_3^3 + y_1^3 x_2^2 y_2 x_3 y_3^2 \\
&\quad + y_1^3 x_2 y_2^2 x_3^2 + y_1^3 y_2^3 x_3^3.
\end{aligned}$$

Definition 42. For even $m = 2k$ we define

$$(29) \quad q_{\mathbf{B}}(x_1, y_1, \dots, x_k, y_k) := (y_1 \cdots y_k)^{2(d_m - d_1 - m) + 3} q_A \left(\frac{x_1}{y_1}, \dots, \frac{x_k}{y_k} \right).$$

For odd $m = 2k - 1$ we set

$$(30) \quad q_{\mathbf{B},k}(x_1, y_1, \dots, x_k, y_k) := (y_1 \cdots y_{k-1})^{2d_m - 2d_1 - 3m + 6} y_k^{d_m - d_1 - m + 1} q_{\mathbf{A},k} \left(\frac{x_1}{y_1}, \dots, \frac{x_k}{y_k} \right).$$

Lemma 43. *Let \mathbf{B} be of the form (28).*

(i) *If $m = 2k$ is even, then*

$$\begin{aligned} & \det D_{c,x} S_{k,\mathbf{B}}(c_1, \dots, c_k, x_1, y_1, \dots, x_k, y_k) \\ &= c_1 \cdots c_k \cdot (x_1 \cdots x_k)^{2d_1} \cdot \prod_{1 \leq i < j \leq k} (x_j y_i - x_i y_j)^4 \cdot q_{\mathbf{B}}(x_1, y_1, \dots, x_k, y_k). \end{aligned}$$

(ii) *If $m = 2k - 1$ is odd, then*

$$\begin{aligned} & \det(D_{c,x} S_{k-1,\mathbf{B}}(c_1, \dots, c_{k-1}, x_1, y_1, \dots, y_{k-1}), s_{\mathbf{B}}(x_k, y_k)) = c_1 \cdots c_{k-1} (x_1 \cdots x_{k-1})^{2d_1} \\ & \times x_k^{d_1} \prod_{1 \leq i < j \leq k-1} (x_j y_i - x_i y_j)^4 \cdot \prod_{i=1}^{k-1} (x_k y_i - x_i y_k)^2 \cdot q_{\mathbf{B},k}(x_1, \dots, x_k). \end{aligned}$$

(iii) $q_{\mathbf{B}}$ in (29) and $q_{\mathbf{B},k}$ in (30) are in $\mathbb{N}_0[x_1, y_1, \dots, x_k, y_k]$.

(iv) $\mathcal{N}_{\mathbf{B}} = \lceil \frac{m}{2} \rceil$.

Proof. (i): Again it suffices to prove the formulas in the case $c_1 = \dots = c_k = 1$. We set $u = \frac{x}{y}$ and $u_i = \frac{x_i}{y_i}$. Using the relation $\partial_x = y^{-1} \partial_u$ and equations (19) and (29) we compute

$$\begin{aligned} & \det D_{c,x} S_{k,\mathbf{B}}(c_1, \dots, c_k, x_1, y_1, \dots, x_k, y_k) \\ &= \det(s_{\mathbf{B}}(x_1, y_1), \partial_x s_{\mathbf{B}}(x_1, y_1), \dots, s_{\mathbf{B}}(x_k, y_k), \partial_x s_{\mathbf{B}}(x_k, y_k)) \\ &= (y_1 \cdots y_k)^{2d_m} \cdot \det(s_{\mathbf{A}}(u_1), \partial_x s_{\mathbf{A}}(u_1), \dots, s_{\mathbf{A}}(u_k), \partial_x s_{\mathbf{A}}(u_k)) \\ &= (y_1 \cdots y_k)^{2d_m - 1} \cdot \det(s_{\mathbf{A}}(u_1), \partial_u s_{\mathbf{A}}(u_1), \dots, s_{\mathbf{A}}(u_k), \partial_u s_{\mathbf{A}}(u_k)) \\ &= (y_1 \cdots y_k)^{2d_m - 1} (u_1 \cdots u_k)^{2d_1} \cdot \prod_{1 \leq i < j \leq k} (u_j - u_i)^4 \cdot q_{\mathbf{A}}(u_1, \dots, u_k) \\ &= (x_1 \cdots x_k)^{2d_1} (y_1 \cdots y_k)^{2(d_m - d_1 - m) + 3} \prod_{1 \leq i < j \leq k} (x_j y_i - x_i y_j)^4 \cdot q_{\mathbf{A}} \left(\frac{x_1}{y_1}, \dots, \frac{x_k}{y_k} \right) \\ &= (x_1 \cdots x_k)^{2d_1} \prod_{1 \leq i < j \leq k} (x_j y_i - x_i y_j)^4 \cdot q_{\mathbf{B}}(x_1, y_1, \dots, x_k, y_k). \end{aligned}$$

(ii): We proceed in a similar manner and derive

$$\begin{aligned} & \det(s_{\mathbf{B}}(x_1, y_1), \partial_x s_{\mathbf{B}}(x_1, y_1), \dots, s_{\mathbf{B}}(x_k, y_k)) \\ &= (y_1 \cdots y_{k-1})^{2d_m} \cdot y_k^{d_m} \cdot \det(s_{\mathbf{A}}(u_1), \partial_x s_{\mathbf{A}}(u_1), \dots, s_{\mathbf{A}}(u_k)) \\ &= (y_1 \cdots y_{k-1})^{2d_m - 1} \cdot y_k^{d_m} \cdot \det(s_{\mathbf{A}}(u_1), \partial_u s_{\mathbf{A}}(u_1), \dots, s_{\mathbf{A}}(u_k)) \\ &= (u_1 \cdots u_{k-1})^{2d_1} u_k^{d_1} (y_1 \cdots y_{k-1})^{2d_m - 1} y_k^{d_m} \prod_{1 \leq i < j \leq k-1} (u_j - u_i)^4 \prod_{i=1}^{k-1} (u_k - u_i)^2 \\ & \quad \times q_{\mathbf{A},k}(u_1, \dots, u_k) \\ &= (x_1 \cdots x_{k-1})^{2d_1} x_k^{d_1} (y_1 \cdots y_{k-1})^{2d_m - 2d_1 - 3m + 6} y_k^{d_m - d_1 - m + 1} \prod_{1 \leq i < j \leq k-1} (x_j y_i - x_i y_j)^4 \\ & \quad \times \prod_{i=1}^{k-1} (x_k y_i - x_i y_k)^2 \cdot q_{\mathbf{A},k} \left(\frac{x_1}{y_1}, \dots, \frac{x_k}{y_k} \right) \end{aligned}$$

$$= (x_1 \cdots x_{k-1})^{2d_1} x_k^{d_1} \prod_{1 \leq i < j \leq k} (x_j y_i - x_i y_j)^4 \cdot \prod_{i=1}^{k-1} (x_k y_i - x_i y_k)^2 \cdot q_{\mathbf{B},k}(x_1, y_1, \dots, x_k, y_k).$$

(iii): First we show that $q_{\mathbf{B}}$ and $q_{\mathbf{B},k}$ are polynomials. That they are polynomials in x_1, \dots, x_k is clear, since $q_{\mathbf{A}}$ and $q_{\mathbf{A},k}$ are polynomials in the coordinates and all x_i appear only with non-negative exponent in the definitions. Therefore, it suffices to show that they are also polynomials in all y_i . We will only prove the statement for $q_{\mathbf{B}}$, for $q_{\mathbf{B},k}$ the same chain of arguments holds.

Assume the contrary. Then $q_{\mathbf{B}}$ contains a term with y_k^{-l} for some $l > 0$ with non-zero coefficient. Let l be the largest such l and let $f(x, y) := \sum_i a_i x^{\alpha_i} y^{\beta_i}$ be the factor of y_k^{-l} in $q_{\mathbf{B}}$. Since f is non-zero by assumption, there are $Z = (x_1, y_1, \dots, x_k, y_k) \in \mathbb{R}^{2k}$ and $\varepsilon > 0$ such that f is non-zero on the ball $B_\varepsilon(Z)$ centered at Z with radius ε .

On the other hand, we expand $\prod_{1 \leq i < j \leq k} (x_j y_i - x_i y_j)^4$ and let $g(x, y) := \sum_i b_i x^{\gamma_i} y^{\delta_i}$ be the sum of all terms therein which contain no y_k . Then g is a polynomial in all x_i and y_i and g is not the zero-polynomial. Hence g is not identically zero on $B_\varepsilon(Z)$ and so is

$$fg = \sum_{i,j} a_i b_j x^{\alpha_i + \gamma_j} y^{\beta_i + \delta_j}.$$

From the Laplace expansion it follows that the determinant

$$(*) \quad \det(s_{\mathbf{B}}(x_1, y_1), \partial_x s_{\mathbf{B}}(x_1, y_1), \dots, s_{\mathbf{B}}(x_k, y_k), \partial_x s_{\mathbf{B}}(x_k, y_k))$$

is a polynomial in x_i, y_j . Further, fg appears in the expansion of the product

$$(**) \quad (x_1 \cdots x_k)^{2e_1} \prod_{1 \leq i < j \leq k} (x_j y_i - x_i y_j)^4 \cdot q_{\mathbf{B}}(x_1, y_1, \dots, x_k, y_k)$$

and by the maximality of l it does not cancel. Hence $(*)$ does not contain a term with y_k^{-l} but $(**)$ does. Since both are equal by (i), we get a contradiction. Thus, $q_{\mathbf{B}}$ is a polynomial in all y_i .

It remains to show that all coefficients of $q_{\mathbf{B}}$ are in \mathbb{N}_0 . Since $q_{\mathbf{A}}$ comes from the Schur polynomial $p_{\mathbf{A}}$ (see (25)), its coefficients are in \mathbb{N}_0 . This is not changed by multiplication with $(y_1 \cdots y_k)^{2(d_m - d_1 - m) + 3}$, so $q_{\mathbf{B}}$ has \mathbb{N}_0 -coefficients as well.

(iv): By (iii) all nonzero coefficients of $q_{\mathbf{B}}$ are positive integers. Hence, by (i) and (ii), we can find real numbers $x_1, y_1, \dots, x_k, y_k$ such that the corresponding determinants are non-zero. Hence $\mathcal{N}_{\mathbf{B}} = k = \lceil \frac{m}{2} \rceil$. \square

Example 44. (1) Let $\mathbf{B} = \{xy^7, x^4y^4, x^7y\}$. Then we have

$$\begin{aligned} & \det(s_{\mathbf{B}}(x_1, y_1), \partial_x s_{\mathbf{B}}(x_1, y_1), s_{\mathbf{B}}(x_2, y_2)) \\ &= 3x_1^4 y_1^5 x_2 y_2 (x_1 y_2 - x_2 y_1)^2 (x_1^2 y_2^2 + x_1 x_2 y_1 y_2 + x_2^2 y_1^2). \end{aligned}$$

(2) $\mathbf{B} = \{xy^5, x^2y^4, x^6\}$. Then

$$\begin{aligned} & \det(s_{\mathbf{B}}(x_1, y_1), \partial_x s_{\mathbf{B}}(x_1, y_1), s_{\mathbf{B}}(x_2, y_2)) \\ &= x_1^2 x_2 y_1^4 (x_1 y_2 - x_2 y_1)^2 (4x_1^3 y_2^3 + 3x_1^2 x_2 y_1 y_2^2 + 2x_1 x_2^2 y_1^2 y_2 + x_2^3 y_1^3). \end{aligned}$$

The following theorem is the main result of this section. It gives sufficient conditions for the validity of formula (32) concerning the Carathéodory number $\mathcal{C}_{\mathbf{B}}$.

Theorem 45. Let $m, d_1, d_2, \dots, d_m, d \in \mathbb{N}$ be such that $0 = d_1 < \dots < d_m = 2d$, put $\mathbf{A} = \{1, x^{d_2}, \dots, x^{d_m}\}$, $\mathbf{B} = \{y^{2d}, x^{d_2} y^{2d-d_2}, \dots, x^{d_{m-1}} y^{2d-d_{m-1}}, x^{2d}\}$, and $\mathcal{Z} := \mathcal{Z}(q_{\mathbf{A}})$ if m is even or $\mathcal{Z} := \mathcal{Z}(q_{\mathbf{A},1}) \cap \dots \cap \mathcal{Z}(q_{\mathbf{A},k})$ if $m = 2k - 1$ is odd, where $q_{\mathbf{A}}$ and $q_{\mathbf{A},j}$ are as in Definition 38. Suppose that

$$(31) \quad (x_1, \dots, x_k) \in \mathcal{Z} \Rightarrow \exists i \neq j : x_i = x_j.$$

Then

$$(32) \quad \mathcal{C}_A = \mathcal{C}_B = \mathcal{N}_A = \mathcal{N}_B = \left\lceil \frac{m}{2} \right\rceil.$$

Proof. Recall that \mathcal{S}_A and \mathcal{S}_B denote the moment cones of A and B, respectively. We set $\partial^* \mathcal{S}_A := \partial \mathcal{S}_A \cap \mathcal{S}_A$.

By Lemmas 40(iii) and 43(iv) we have $\mathcal{N}_A = \mathcal{N}_B = \left\lceil \frac{m}{2} \right\rceil$. Further, $\mathcal{N}_A \leq \mathcal{C}_A$ and $\mathcal{N}_A \leq \mathcal{C}_B$ by Theorem 27. Therefore, it suffices to show that $\mathcal{C}_A \leq \mathcal{N}_A$ and $\mathcal{C}_B \leq \mathcal{N}_A$.

First we prove that $\mathcal{C}_B \leq \mathcal{N}_A$.

Let $s \in \mathcal{S}_B$. Since $\mathcal{X} = \mathbb{P}(\mathbb{R})$ is compact and condition (6) is satisfied (with $e(x, y) := x^{2d} + y^{2d} \in \mathcal{B}$), Proposition 8 applies with $x = (1, 0)$. Hence the supremum $c_s(1, 0) := \sup \{c \in \mathbb{R} : s - cs_B(1, 0) \in \mathcal{S}_B\}$ is attained and $s' := s - c_s(1, 0)s_B(1, 0) \in \partial \mathcal{S}_B$. By Proposition 30(iii) all representing measures (C', X') of s' are singular. They do not contain $(1, 0)$ as an atom. (Indeed, otherwise $c_s(1, 0)$ could be increased which contradicts to the maximality of $c_s(1, 0)$.) Since the polynomials of B are homogeneous, we can assume without loss of generality that $X'_i = (x'_i, 1)$ with x'_i pairwise different, say $x'_1 < x'_2 < \dots < x'_l$, and $s' \in \partial^* \mathcal{S}_A$, i.e., s' is a boundary moment sequence of \mathcal{S}_A . But from (31) and Lemma 43(i) and (ii), it follows that $l < \mathcal{N}_A$, that is, $\mathcal{C}_B(s) \leq l + 1 \leq \mathcal{N}_B = \mathcal{N}_A$. This completes the proof of the inequality $\mathcal{C}_B \leq \mathcal{N}_A$.

Next we show that $\mathcal{C}_A \leq \mathcal{N}_A$.

If $s \in \partial^* \mathcal{S}_A$, then $\mathcal{C}_A(s) < \mathcal{N}_A$ by the preceding proof. Now let $s \in \text{int } \mathcal{S}_A = \text{int } \mathcal{S}_B$. Then $\mathcal{C}_B(s) \leq \mathcal{N}_A$ by the preceding paragraph and it suffices to show that s has an at most \mathcal{N}_A -atomic representing measure which does not have an atom at $(1, 0)$. We choose $\varepsilon > 0$ such that $B_\varepsilon(s) \subseteq \text{int } \mathcal{S}_A$.

Let $c_t(x)$ be defined by (7). Since $t \mapsto L_t$ is a continuous map of $\mathbb{R}^m \rightarrow \mathcal{A}^*$, $t \mapsto L_t(e)$ is continuous. Hence, $c_t(x) \leq e(x)^{-1} L_t(e)$ (by Proposition 8) is bounded from above on $\overline{B_\varepsilon(s)}$. Then the supremum C of $c_t(1, 0)$ on $\overline{B_\varepsilon(s)}$ is finite. Let

$$T := \bigcup_{c \in [0, C+1]} \overline{B_\varepsilon(s - c \cdot s_B(1, 0))}$$

be the ε -tube around the line $\gamma := s - [0, C+1] \cdot s_B(1, 0)$. Write $T = T_1 \cup T_2 \cup T_3$ with $T_2 := T \cap \partial \mathcal{S}_B$, $T_1 := T \cap \text{int } \mathcal{S}_B$, and $T_3 := T \setminus (T_1 \cup T_2)$, i.e., T_1 is the part inside \mathcal{S}_B , T_3 is the part outside \mathcal{S}_B , and T_2 is the boundary part of \mathcal{S}_B in T . Since \mathcal{S}_B is closed and convex, T_2 is closed and every path in T starting in T_1 and ending in T_3 contains at least one point in T_2 . By construction, $t' := t - c_t(1, 0)s_B(1, 0) \in T_2$ for all $t \in T_1$ and no representing measure of t' contains $(1, 0)$ as an atom, i.e., $T_2 \subset \partial^* \mathcal{S}_A$. Then $\gamma = s - [0, 1] \cdot (C+1)s_B(1, 0) \subset T$ and $s \in \gamma \cap T_1$ and $s - (C+1)s_B(1, 0) \in \gamma \cap T_3$, so that $s' = s - c_s(1, 0)s_B(1, 0) \in T_2$. Since s_B is continuous and $C < \infty$ there is a $\delta > 0$ such that

$$\|(s - (C+1)s_B(1, 0)) - (s - (C+1)s_B(1, \delta))\| = (C+1)\|s_B(1, 0) - s_B(1, \delta)\| < \varepsilon.$$

Thus, $s - (C+1)s_B(1, \delta) \in T_3$. Then $\gamma_\delta := s - [0, 1] \cdot (C+1)s_B(1, \delta) \subset T$ and $s \in \gamma_\delta \cap T_1$ and $s - (C+1)s_B(1, \delta) \in \gamma_\delta \cap T_3$, i.e.,

$$s'_\delta = s - c_s(1, \delta)s_B(1, \delta) \in T_2 \subset \partial^* \mathcal{S}_A.$$

Summarizing, $s = s'_\delta + c_s(1, \delta)s_B(1, \delta)$ and s'_δ has a k -atomic representing measure ($k < \mathcal{N}_A$) which has no atom at $(1, 0)$. Therefore, s has an l -atomic presenting measure ($l \leq \mathcal{N}_A$) which has no atom at $(1, 0)$. This proves $\mathcal{C}_A(s) \leq \mathcal{N}_A$. \square

We illustrate the preceding by the following examples.

Example 46. Let $A = \{1, x^2, x^3, x^5, x^6\}$ and $B = \{y^6, x^2y^4, x^3y^3, x^5y, x^6\}$, that is, $m = 5$. Then we have

$$(33) \quad \det(s_A(x), s'_A(x), s_A(y), s'_A(y), s_A(z)) = (x-y)^4(x-z)^2(y-z)^2 \cdot f(x, y, z),$$

where

$$(34) \quad f(x, y, z) := xy(x^3y + 4x^2y^2 + xy^3 + 2x^3z + 10x^2yz + 10xy^2z + 2y^3z + 4x^2z^2 + 7xyz^2 + 4y^2z^2).$$

This implies $\mathcal{N}_B = \mathcal{N}_A = 3$ as also proved in Lemma 35 and 43. Hence $\mathcal{C}_A \geq 3$. From the Richter–Tchakaloff Theorem (Proposition 1) we find $\mathcal{C}_A \leq m = 5$, while Theorem 13 gives a better bound $\mathcal{C}_A \leq m - 1 = 4$.

To apply Theorem 45 we have to check that the assumptions are satisfied. Clearly, $d_1 = 0$ and $d_m = 6$ is even. It remains to show that (31) is true. By symmetry it suffices to verify (31) for

$$f_1(x, y, z) := f(x, y, z), \quad f_2(x, y, z) := f(y, z, x), \quad \text{and} \quad f_3(x, y, z) := f(z, x, y).$$

Set $\mathcal{Z} := \mathcal{Z}(f_1) \cap \mathcal{Z}(f_2) \cap \mathcal{Z}(f_3)$ and let $X = (x, y, z) \in \mathcal{Z}$. If $X = 0$, then (31) holds. Now let $X \neq 0$. Since f is homogeneous, we can scale X such that $x^2 + y^2 + z^2 = 1$. Then we derive (for instance, by using spherical coordinates)

$$(35) \quad \mathcal{Z}(f_1) \cap \mathcal{Z}(f_2) \cap \mathcal{Z}(f_3) = \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\},$$

so (31) is fulfilled. Therefore, by Theorem 45 we have $\mathcal{C}_A = 3$.

A nice application of the preceding example is the following corollary.

Corollary 47. *Let $p(x) = a + bx^2 + cx^3 + dx^5 + ex^6$ be a non-negative polynomial which is not the zero polynomial. Then p has at most 2 distinct real zeros.*

Proof. Assume to the contrary that p has three distinct zeros, say x, y, z . Let s be the moment sequence of the measure $\mu = \delta_x + \delta_y + \delta_z$. Then $L_s(p) = 0$, so s is a boundary point of the moment cone. But from (35) it follows that the determinant (33) is non-zero, so s is an inner point, a contradiction. \square

In the following example the assumption (31) of Theorem 45 is not satisfied and the assertion (32) does not hold.

Example 48. Let $A = \{1, x, x^2, x^6\}$ and $B = \{y^6, xy^5, x^2y^4, x^6\}$. From Theorem 13, $\mathcal{C}_B \leq m - 1 = 3$, while Theorem 27 and

$$\det(s_A(x), s'_A(x), s_A(y), s'_A(y)) = 2(y - x)^4(x + y)(2x^2 + xy + 2y^2)$$

yield $2 = \mathcal{N}_B \leq \mathcal{C}_A$, so that $\mathcal{C}_B \in \{2, 3\}$. We prove that $\mathcal{C}_B = 3$. Let $\nu := \frac{1}{4}(\delta_{-2} + \delta_{-1} + \delta_1 + \delta_2)$. Then

$$s = (s_0, s_1, s_2, s_6)^T = (s_A(-2) + s_A(-1) + s_A(1) + s_A(2))/4 = (1, 0, 2.5, 32.5)^T.$$

By some straightforward computations it can be shown that s has no k -atomic representing measure with $k \leq 2$. Therefore, since $\mathcal{C}_B \in \{2, 3\}$, we have $\mathcal{C}_B = 3 \neq \lceil \frac{3}{2} \rceil$.

Note that $\mathcal{N}_B = 2 = \lceil \frac{3}{2} \rceil$. Thus, the equality (32) fails.

5. CARATHÉODORY NUMBERS: MULTIDIMENSIONAL MONOMIAL CASE

Definition 49. For $n, d \in \mathbb{N}$ set

$$(36) \quad A_{n,d} := \{x^\alpha : \alpha \in \mathbb{N}_0^n, |\alpha| \leq d\},$$

$$(37) \quad B_{n,d} := \{x^\alpha : \alpha \in \mathbb{N}_0^n, |\alpha| = d\}.$$

Note that $|A_{n,d}| = \binom{n+d}{d}$ and $|B_{n,d}| = \binom{n+d-1}{d}$.

Throughout this section, we assume the following: For the polynomials $\mathcal{A}_{n,d} := \text{Lin } A_{n,d}$ we consider the truncated moment problem on $\mathcal{X} = \mathbb{R}^n$, while for the homogeneous polynomials $\mathcal{B}_{n,d} := \text{Lin } B_{n,d}$ the moment problem is treated on the real projective space $\mathcal{X} := \mathbb{P}(\mathbb{R}^{n-1})$. Let S^{n-1} be the unit sphere in \mathbb{R}^n , $n \geq 2$, and S_+^{n-1} the set of points $x \in S^{n-1}$ for which the first non-vanishing coordinate is positive. We consider S_+^{n-1} as a realization of the projective space $\mathbb{P}(\mathbb{R}^{n-1})$.

The following simple fact is often used without mention: A polynomial of $\mathcal{B}_{n,2d}$ is non-negative on S_+^{n-1} , equivalently on $\mathbb{P}(\mathbb{R}^{n-1})$, if and only if it is on \mathbb{R}^{n-1} .

The following example shows how differential geometric methods can be used for the truncated moment problem.

Example 50. Let $n = d = k = 2$, $x^\alpha = (x^{(1)})^{\alpha_1}(x^{(2)})^{\alpha_2}$ and

$$\begin{aligned} \mathbf{A}_{2,2} &= \{x^\alpha : \alpha \in \mathbb{N}_0^2, |\alpha| \leq 2\} \\ &= \{x^\alpha : \alpha = (0,0), (0,1), (1,0), (2,0), (1,1), (0,2)\}. \end{aligned}$$

Then

$$DS_{2,\mathbf{A}}(C, X) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ x_1^{(1)} & c_1 & 0 & x_2^{(1)} & c_2 & 0 \\ x_1^{(2)} & 0 & c_1 & x_2^{(2)} & 0 & c_2 \\ (x_1^{(1)})^2 & 2c_1x_1^{(1)} & 0 & (x_2^{(1)})^2 & 2c_2x_2^{(1)} & 0 \\ x_1^{(1)}x_1^{(2)} & c_1x_1^{(2)} & c_1x_1^{(1)} & x_2^{(1)}x_2^{(2)} & c_2x_2^{(2)} & c_2x_2^{(1)} \\ (x_1^{(2)})^2 & 0 & 2c_1x_1^{(2)} & (x_2^{(2)})^2 & 0 & 2c_2x_2^{(2)} \end{pmatrix},$$

where $C = (c_1, c_2)$ and $X = (x_1, x_2)$, $x_i = (x_i^{(1)}, x_i^{(2)})$. From this we find that

$$\ker DS_{2,\mathbf{A}}(C, X) = \mathbb{R} \cdot v(C, X) \quad \text{with} \quad v(C, X) := \begin{pmatrix} -2 \\ c_1^{-1}(x_1^{(1)} - x_2^{(1)}) \\ c_1^{-1}(x_1^{(2)} - x_2^{(2)}) \\ 2 \\ c_2^{-1}(x_1^{(1)} - x_2^{(1)}) \\ c_2^{-1}(x_1^{(2)} - x_2^{(2)}) \end{pmatrix}.$$

Hence $\text{rank } DS_{2,\mathbf{A}_{2,2}} = 5$ at each point (x_1, x_2) , $x_1 \neq x_2$, so the local rank theorem of differential geometry applies. Fix (C, X) as above. The local rank theorem [Hil03, Proposition 1, p. 309] implies that there is a one-parameter family $(C(t), X(t))$ which has the same moments as (C, X) satisfying the differential equations $\dot{\gamma}(t) = v(C(t), X(t))$ with initial condition $(C(0), X(0)) = (C, X)$. This system is

$$\begin{aligned} \dot{c}_1 &= -2 & \dot{c}_2 &= 2 \\ c_1 \cdot \dot{x}_1^{(1)} &= x_1^{(1)} - x_2^{(1)} & c_2 \cdot \dot{x}_2^{(1)} &= x_1^{(1)} - x_2^{(1)} \\ c_1 \cdot \dot{x}_1^{(2)} &= x_1^{(2)} - x_2^{(2)} & c_2 \cdot \dot{x}_2^{(2)} &= x_1^{(2)} - x_2^{(2)} \end{aligned}$$

and its solution is given by

$$\begin{aligned} c_1(t) &= c_{1,0} - 2t & x_1^{(i)}(t) &= \gamma_{1,i} + \frac{\gamma_{2,i}}{c_{1,0} + c_{2,0}} \cdot \sqrt{\frac{c_{2,0} + 2t}{c_{1,0} - 2t}} \\ c_2(t) &= c_{2,0} + 2t & x_2^{(i)}(t) &= \gamma_{2,i} - \frac{\gamma_{2,i}}{c_{1,0} + c_{2,0}} \cdot \sqrt{\frac{c_{1,0} - 2t}{c_{2,0} + 2t}} \end{aligned}$$

with $t \in (-\frac{c_{2,0}}{2}, \frac{c_{1,0}}{2})$. Here $C = (c_{1,0}, c_{2,0})$ and $X = ((\gamma_{1,1}, \gamma_{1,2}), (\gamma_{2,1}, \gamma_{2,2}))$ are the initial values at $t = 0$. It should be noted that the corresponding moment sequence is indeterminate, but it is a boundary point of the moment cone.

Recall that $\mathcal{N}_{\mathbf{A}} \leq \mathcal{C}_{\mathbf{A}}$ by Theorem 27. There are various other lower bounds for Carathéodory numbers in the literature, see e.g. [DR84, p. 366]. In the case $\mathbf{A}_{2,2k-1}$, M. Möller [Möl76] obtained the lower bound

$$\text{Mö}(2, 2k - 1) := \binom{k+1}{2} + \left\lfloor \frac{k}{2} \right\rfloor.$$

The following result improves Möller's lower bound.

Proposition 51.

$$(38) \quad \text{Mö}(2, 2k-1) \leq \left\lceil \frac{|A_{2,2k-1}|}{3} \right\rceil \leq \mathcal{N}_{A_{2,2k-1}} \leq \mathcal{C}_{A_{2,2k-1}} \quad \text{for } k \in \mathbb{N}.$$

For $k \geq 4$ we have

$$(39) \quad \left\lceil \frac{|A_{2,2k-1}|}{3} \right\rceil - \text{Mö}(2, 2k-1) \geq \frac{(k-2)^2 - 4}{6}.$$

Proof. The second inequality of (38) has been stated in Proposition 23. It remains to prove the first inequality of (38). In the cases $k = 1, 2, 3$ it is verified by direct computations; we omit the details. For $k \geq 4$ it follows from the following computation:

$$\begin{aligned} \left\lceil \frac{1}{3} \binom{2k+1}{2} \right\rceil - \left(\binom{k+1}{2} + \left\lfloor \frac{2}{k} \right\rfloor \right) &\geq \frac{1}{3} \binom{2k+1}{2} - \left(\binom{k+1}{2} + \frac{k}{2} \right) \\ &= \frac{(2k+1)k}{3} - \frac{(k+1)k}{2} - \frac{k}{2} = \frac{(k-2)^2 - 4}{6}. \quad \square \end{aligned}$$

Before we turn to our next result we restate the Alexander–Hirschowitz Theorem [AH95]. We denote by $V_{n,d,r}$ the vector space of polynomials in n variables of degree at most d having singularities at r general points in \mathbb{R}^n .

Proposition 52. *The subspace $V_{n,d,r}$ has the expected codimension*

$$\min \left(r(n+1), \binom{n+d}{d} \right)$$

except for the following cases:

- (i) $d = 2; 2 \leq r \leq n, \text{codim } V_{n,2,r} = r(n+1) - r(r-1)/2;$
- (ii) $d = 3; n = 4, r = 7, \dim V_{4,3,7} = 1;$
- (iii) $d = 4; (n, r) = (2, 5), (3, 9), (4, 14), \dim V_{n,4,r} = 1.$

Theorem 53. *We have*

$$(40) \quad \mathcal{N}_{A_{n,d}} = \left\lceil \frac{1}{n+1} \binom{n+d}{n} \right\rceil,$$

except for the following cases

- (i) $d = 2: \mathcal{N}_{A_{n,2}} = n + 1.$
- (ii) $n = 4, d = 3: \mathcal{N}_{A_{4,3}} = 8.$
- (iii) $n = 2, d = 4: \mathcal{N}_{A_{2,4}} = 6.$
- (iv) $n = 3, d = 4: \mathcal{N}_{A_{3,4}} = 10.$
- (v) $n = d = 4: \mathcal{N}_{A_{4,4}} = 15.$

Proof. From the corresponding definitions of $V_{n,d,r}$ and $DS_{k,A_{n,d}}$ we obtain

$$\text{codim } V_{n,d,r} = |A_{n,d}| - \dim V_{n,d,r} = \text{rank } DS_{k,A_{n,d}}((1, \dots, 1), X).$$

Therefore, apart from exceptional cases, (40) follows at once from the Alexander–Hirschowitz Theorem. Next we treat the exceptions.

(i): Note that $\mathcal{N}_{A_{n,2}} \geq \lceil n/2 \rceil + 1$. Since for all k satisfying $\lceil n/2 \rceil + 1 \leq k \leq n$ the matrix $DS_{k,A_{n,2}}((1, \dots, 1), X)$ has not the expected full rank for any X , the first k with full rank is $k = n + 1$.

(ii): Since $\mathcal{N}_{A_{4,3}} \geq 7$ and $DS_{7,A_{4,3}}((1, \dots, 1), X)$ has not the expected full rank for any X , $\mathcal{N}_{A_{4,3}} = 8$.

(iii): We have $\mathcal{N}_{A_{2,4}} \geq 5$ and $DS_{5,A_{2,4}}((1, \dots, 1), X)$ has not the expected full rank for any X . Hence $\mathcal{N}_{A_{2,4}} = 6$.

(iv): Then $\mathcal{N}_{A_{3,4}} \geq 9$ and $DS_{9,A_{3,4}}((1, \dots, 1), X)$ has not the expected full rank for any X . Thus $\mathcal{N}_{A_{3,4}} = 10$.

(v): Then $\mathcal{N}_{\mathbf{A}_{4,4}} \geq 14$ and $DS_{14,\mathbf{A}_{4,4}}((1, \dots, 1), X)$ has not the expected full rank for any X . Therefore, $\mathcal{N}_{\mathbf{A}_{4,4}} = 15$. \square

For the homogeneous case we have

Corollary 54. $\mathcal{N}_{\mathbf{B}_{n+1,d}} = \mathcal{N}_{\mathbf{A}_{n,d}}$.

Proof. Let $X = (X_1, \dots, X_k) \in \mathbb{R}^{nk}$, $k = \mathcal{N}_{\mathbf{A}_{n,d}}$, be such that $DS_{k,\mathbf{A}_{n,d}}(1, X)$ has full rank. Then $DS_{k,\mathbf{B}_{n+1,d}}(1, Y)$ with $Y = ((X_1, 1), \dots, (X_k, 1))$ has full rank, so that $\mathcal{N}_{\mathbf{A}_{n,d}} = k \geq \mathcal{N}_{\mathbf{B}_{n+1,d}}$.

On the other hand, let $Y = (Y_1, \dots, Y_k) \in \mathbb{R}^{(n+1)k}$, $k = \mathcal{N}_{\mathbf{B}_{n+1,d}}$, be such that $DS_{k,\mathbf{B}_{n+1,d}}(1, Y)$ has full rank. We can assume that all $(n+1)$ -th coordinates of Y_i are non-zero by the continuity of the determinant and therefore they can be chosen to be 1, since we are in $\mathbb{P}(\mathbb{R}^n)$. The column $\partial_{n+1}s_{\mathbf{B}_{n+1,d}}(Y_i)$ depends linearly on $s_{\mathbf{B}_{n+1,d}}(Y_i)$ and $\partial_j s_{\mathbf{B}_{n+1,d}}(Y_i)$, $j = 1, \dots, n$. Therefore, omitting this column does not change the rank. Hence $DS_{k,\mathbf{A}_{n,d}}(1, X)$ with $Y_i = (X_i, 1)$ has full rank, that is, $\mathcal{N}_{\mathbf{A}_{n,d}} \leq k = \mathcal{N}_{\mathbf{B}_{n+1,d}}$. \square

6. CARATHÉODORY NUMBERS AND ZEROS OF POSITIVE POLYNOMIALS

For $f \in \mathcal{B}_{3,2d}$, $\mathcal{Z}_{\mathbb{P}}(f)$ denotes the projective zero set of f . Set

$$(41) \quad \alpha(2d) := \frac{3}{2}d(d-1) + 1.$$

In this section we use the following proposition of Choi, Lam, and Reznick [CLR80].

Proposition 55. *Let $f \in \mathcal{B}_{3,2d}$. Suppose that $f \in \text{Pos}(\mathbb{R}^3)$ and $|\mathcal{Z}_{\mathbb{P}}(f)| > \alpha(2d)$. Then $|\mathcal{Z}_{\mathbb{P}}(f)|$ is infinite and there are polynomials $p \in \mathcal{B}_{3,2d_1}$, $q \in \mathcal{B}_{3,2d_2}$ such that $f = pq^2$, where $d_1 + d_2 = d$, $p \in \text{Pos}(\mathbb{R}^3)$, $|\mathcal{Z}_{\mathbb{P}}(p)| < \infty$, q is indefinite, and $|\mathcal{Z}_{\mathbb{P}}(q)|$ is infinite. (It is possible that p is a positive real constant; in this case $d_1 = 0$ and we set $\mathcal{B}_{3,0} := \mathbb{R}$.)*

The main aim of this section is to derive *upper bounds* for the Carathéodory number $\mathcal{C}_{\mathbf{B}_{n,2d}}$, $n = 3$. The first approach (Theorem 57) applies also to cases with $n > 3$ (see Theorem 59). The second approach (Theorem 62) is based on Bezout's Theorem and gives better bounds.

For $d \in \mathbb{N}$ let $\beta(2d)$ denote the maximum of $|\mathcal{Z}_{\mathbb{P}}(f)|$, where $f \in \mathcal{B}_{3,2d}$, $f \in \text{Pos}(\mathbb{R}^2)$ and $\mathcal{Z}_{\mathbb{P}}(f)$ is finite. By the Choi–Lam–Reznick Theorem (Proposition 55), $\beta(d) \leq \alpha(d)$ for $d \in \mathbb{N}$. We abbreviate $\mathcal{C}_{2d} := \mathcal{C}_{\mathbf{B}_{3,2d}}$.

Theorem 56.

$$(42) \quad \mathcal{C}_{2d} \leq \max_{k=0,\dots,d} \left\{ \binom{2d+2}{2} - \binom{2d+2-k}{2} + \beta(2(d-k)) \right\} + 1.$$

Proof. Let $s \in \mathcal{S}$. Since the projective space $\mathbb{P}(\mathbb{R}^{n-1})$ is compact and condition (6) holds with $e := x_1^{2d} + x_2^{2d} + x_3^{2d}$, it follows from Proposition 8 that $\mathcal{C}_{2d} \leq \max_{s \in \partial \mathcal{S}} \mathcal{C}_{2d}(s) + 1$. Therefore, it is sufficient to show

$$(*) \quad \mathcal{C}_{2d}(s) \leq \max_{k=0,\dots,d} \left\{ \binom{2d+2}{2} - \binom{2d+2-k}{2} + \beta(2(d-k)) \right\}$$

for all $s \in \partial \mathcal{S}$.

Let $s = \sum_{i=1}^l c_i s_{\mathbf{B}_{3,2d}}(x_i)$ be an l -atomic representing measure of $s \in \partial \mathcal{S}$. Since $s \in \partial \mathcal{S}$, there exists a polynomial $p \in \mathcal{B}_{3,2d}$, $p \neq 0$, such that $p(x) \geq 0$ on $\mathbb{P}(\mathbb{R}^2)$ and $L_s(p) = 0$. Then $\text{supp } \mu \subseteq \mathcal{Z}(p)$, that is, $x_1, \dots, x_l \in \mathcal{Z}(p)$.

We can assume without loss of generality that the set $\{s_{\mathbf{B}_{3,2d}}(x_i)\}_{i=1,\dots,l}$ is linearly independent. Indeed, assume that these vectors are linearly dependent and

let $\sum_{i=1}^l d_i s_{\mathcal{B}_{3,2d}}(x_i) = 0$ be a non-trivial linear combination. Since all $c_i > 0$, there exists $\varepsilon > 0$ such that $c_i + \varepsilon d_i \geq 0$ for all i and $c_j + \varepsilon d_j = 0$ for one j . Hence $\mu' = \sum_{i=1}^l (c_i + \varepsilon d_i) \cdot s_{\mathcal{B}_{3,2d}}(x_i)$ is a $(l-1)$ -atomic representing measure of s .

The polynomial $p \in \mathcal{B}_{3,2d}$ is non-negative on $\mathbb{P}(\mathbb{R}^2)$, hence on \mathbb{R}^3 , so the Choi–Lam–Reznick Theorem (Proposition 55) applies. There are two cases:

- a) $|\mathcal{Z}(p)| \leq \beta(2d)$.
- b) $p = h^2 q$, where $k := \deg(h) \geq 1$.

In the case a) we have $|\mathcal{Z}(p)| \leq \beta(2d)$ by the definition of $\beta(2d)$ and therefore $\mathcal{C}_{2d}(s) \leq \beta(2d)$. This is the case $d = k$ in (*).

Now we turn to case b). Then $k = 1, \dots, d-1$. Let $D(k)$ denote the largest l for which there exist $y_1, \dots, y_l \in \mathcal{Z}(h)$ such that the vector $s_{\mathcal{B}_{3,2d}}(y_1), \dots, s_{\mathcal{B}_{3,2d}}(y_l)$ are linearly independent. Then, by the paragraph before last, we have

$$(43) \quad \mathcal{C}_{2d}(s) \leq D(k) + \beta(2(d-k)).$$

Let $y_1, \dots, y_l \in \mathcal{Z}(h)$. We define

$$M(y_1, \dots, y_l) := \begin{pmatrix} s_{\mathcal{B}_{3,2d}}(y_1)^T \\ \vdots \\ s_{\mathcal{B}_{3,2d}}(y_l)^T \end{pmatrix}$$

and $h_\alpha := x^\alpha h$ for $\alpha \in \mathbb{N}_0^3, |\alpha| = 2d-k$. Let \tilde{h}_α be the coefficient vector of h_α , that is, $h_\alpha(\cdot) = \langle \tilde{h}_\alpha, s_{\mathcal{B}_{3,2d}}(\cdot) \rangle$. Since $s_{\mathcal{B}_{3,2d}}(y_i)^T \cdot \tilde{h}_\alpha = \langle \tilde{h}_\alpha, s_{\mathcal{B}_{3,2d}}(y_i) \rangle = y_i^\alpha h(y_i) = 0$, we have $\tilde{h}_\alpha \in \ker M(y_1, \dots, y_l)$. Clearly, the vectors \tilde{h}_α are linearly independent. Therefore, using (43) we derive

$$\begin{aligned} \mathcal{C}_{2d}(s) &\leq D(k) + \beta(2(d-k)) \\ &\leq \max \text{rank } M(y_1, \dots, y_l) + \beta(2(d-k)) \leq |\mathcal{B}_{3,2d}| - |\mathcal{B}_{3,2d-k}| + \beta(2(d-k)) \\ &= \binom{2d+2}{2} - \binom{2d+2-k}{2} + \beta(2(d-k)) \end{aligned}$$

which is the k -th term in (*).

Summarizing, we have $k = d$ in case a) and $k = 1, \dots, d-1$ in case b). Thus we have proved (*) for arbitrary $s \in \partial \mathcal{S}$ which completes the proof. \square

As far as the authors know, the numbers $\beta(2d)$ are not yet known for $d \geq 4$, but we have $\beta(2d) \leq \alpha(2d)$ by Proposition 55.

Theorem 57. *For $d \in \mathbb{N}$ we have*

$$(44) \quad \mathcal{C}_{2d} \leq \alpha(2(d+1)) = \frac{3}{2}d(d+1) + 1.$$

Proof. Since $\beta(2d) \leq \alpha(2d) = \frac{3}{2}d(d-1) + 1$ and $(d-k)(k+3) - 1 \geq 0$ for all $d \in \mathbb{N}$ and $k = 0, \dots, d-1$, we have for (42)

$$\begin{aligned} \frac{3}{2}d(d+1) &= \binom{2d+2}{2} - \binom{d+2}{2} \\ &= \binom{2d+2}{2} - \binom{2d-k+2}{2} + \alpha(d-k) + (d-k)(k+3) - 1 \\ &\geq \binom{2d+2}{2} - \binom{2d-k+2}{2} + \alpha(d-k). \end{aligned}$$

Inserting the latter into (42) we obtain the assertion. \square

In Table 1 we collect some numerical cases of Carathéodory bounds.

The next proposition is also due to Choi–Lam–Reznick [CLR80]. We will use it to derive a bound for the Carathéodory number $\mathcal{C}_{\mathcal{B}_{4,4}}$.

$2d$	Lower Bounds $\mathcal{N}_{\mathcal{B}_{3,2d}}$	Upper Bounds for \mathcal{C}_{2d} from				known \mathcal{C}_{2d}
		Prop. 1	Thm. 13	Thm. 57	Thm. 62	
2	3	6	5	4	4	3 [Rez92]
4	6	15	14	10	8	6 [Rez92]
6	10	28	27	19	14	11 [Kun14]
8	15	45	44	31	22	—
10	22	66	65	46	32	—
12	31	91	90	64	47	—
14	40	120	119	85	65	—
16	51	153	152	109	86	—
18	64	190	189	136	110	—
20	77	231	230	166	137	—
40	287	861	860	631	572	—
100	1717	5151	5150	3826	3677	—
1000	167167	501501	501500	375751	374252	—

TABLE 1. Bounds on the Carathéodory numbers \mathcal{C}_{2d} for $d = 1, \dots, 10, 20, 50, 500$ from Proposition 1 and Theorems 13, 57, 62.

Proposition 58. *If $p \in \mathcal{B}_{4,4}$ and $|\mathcal{Z}(p)| > 11$, then p is a sum of at most six squares of quadratics.*

Theorem 59. $\mathcal{C}_{\mathcal{B}_{4,4}} \leq 26$.

Proof. Let s be a boundary moment sequence. Then there exists $p \in \mathcal{B}_{4,4}, p \neq 0$, such that $p \in \text{Pos}(\mathbb{R}^3)$ and $L_s(p) = 0$. By Proposition 58, $|\mathcal{Z}(p)| \leq 11$ or we have $p = f_1^2 + \dots + f_6^2$ for some $f_1, \dots, f_6 \in \mathcal{B}_{4,2}$. In the following proof we give an upper bound on the maximal number l of linearly independent vectors $s_{\mathcal{B}_{4,4}}(x_1), \dots, s_{\mathcal{B}_{4,4}}(x_l)$ with $x_i \in \mathcal{Z}(p)$. By Theorem 18, this number l is an upper bound of $\mathcal{C}_{\mathcal{B}_{4,4}}(s)$. We proceed in a similar manner as in the proof of Theorem 57.

By Proposition 58 we have two cases:

- a) $|\mathcal{Z}(p)| \leq 11$,
- b) $p = f_1^2 + \dots + f_k^2, k \leq 6$.

In the case a) we clearly have $l \leq |\mathcal{Z}(p)| \leq 11$.

Now we treat case b). Clearly, $\mathcal{Z}(f_1^2 + \dots + f_k^2) \subseteq \mathcal{Z}(f_1^2) = \mathcal{Z}(f_1)$. Hence it suffices to determine the maximal number l for a single square $p = f^2$, where $f \in \mathcal{B}_{4,2}, f \neq 0$. Let $x_1, \dots, x_l \in \mathcal{Z}(f)$ be such that the set $\{s_{\mathcal{B}_{4,4}}(x_i)\}_{i=1, \dots, l}$ is linearly independent. Define

$$M(x_1, \dots, x_l) := \begin{pmatrix} s_{\mathcal{B}_{4,4}}(x_1)^T \\ \vdots \\ s_{\mathcal{B}_{4,4}}(x_l)^T \end{pmatrix},$$

$f_\alpha := x^\alpha f$ for $\alpha \in \mathbb{N}_0^4, |\alpha| = 2$, and \tilde{f}_α by $f_\alpha(\cdot) = \langle \tilde{f}_\alpha, s_{\mathcal{B}_{4,4}}(\cdot) \rangle$. Then we have $\tilde{f}_\alpha \in \ker M(x_1, \dots, x_l)$, since $s_{\mathcal{B}_{4,4}}(x_i)^T \cdot \tilde{f}_\alpha = \langle \tilde{f}_\alpha, s_{\mathcal{B}_{4,4}}(x_i) \rangle = x_i^\alpha f(x_i) = 0$. The vectors \tilde{f}_α are linearly independent. Therefore, $\dim \ker M(x_1, \dots, x_l) \geq \#f_\alpha = |\mathcal{B}_{4,2}|$ and

$$l = \text{rank } M(x_1, \dots, x_l) \leq |\mathcal{B}_{4,4}| - |\mathcal{B}_{4,2}| = \binom{7}{3} - \binom{5}{3} = 25.$$

This proves that the moment sequence s can be represented by at most 25 atoms.

Summarizing both cases, we have shown that each $s \in \partial\mathcal{S}$ has a k -atomic representing measure with $k \leq 25$. Therefore, by Proposition 8, $\mathcal{C}_{\mathcal{B}_{4,4}} \leq 25 + 1 = 26$. \square

Proposition 1 yields $\mathcal{C}_{\mathcal{B}_{4,4}} \leq 35$, while Theorem 13 gives $\mathcal{C}_{\mathcal{B}_{4,4}} \leq 34$. Combining the upper bound of Theorem 59 with the lower bound from Theorem 27 we get

$$(45) \quad \mathcal{N}_{\mathcal{B}_{4,4}} = \mathcal{N}_{\mathcal{A}_{3,4}} = 10 \leq \mathcal{C}_{\mathcal{B}_{4,4}} \leq 26.$$

Now we give another approach to obtain estimates of the Carathéodory number $\mathcal{C}_{\mathcal{B}_{3,2d}}$ from above. It is based on Bezout's Theorem.

Let $f_1 \in \mathcal{B}_{3,d_1}$ and $f_2 \in \mathcal{B}_{3,d_2}$. For each point $t \in \mathcal{Z}_{\mathbb{P}}(f_1) \cap \mathcal{Z}_{\mathbb{P}}(f_2)$ the *intersection multiplicity* $I_t(f_1, f_2) \in \mathbb{N}$ of the projective curves $f_1 = 0$ and $f_2 = 0$ at t is defined in [Wal78, III, Section 2.2]. We do not restate the precise definition here. In what follows we use only the fact that $I_t(f_1, f_2) \geq 2$ if t is a singular point of one of the curves $f_1 = 0$ or $f_2 = 0$.

We use the following version of *Bezout's Theorem*. The symbol $|Z|$ denotes the number of points of a set Z .

Lemma 60. *If $f_1 \in \mathcal{B}_{3,d_1}$ and $f_2 \in \mathcal{B}_{3,d_2}$ are relatively prime in $\mathbb{R}[x_1, x_2, x_3]$, then*

$$\sum_{t \in \mathcal{Z}_{\mathbb{P}}(f_1) \cap \mathcal{Z}_{\mathbb{P}}(f_2)} I_t(f_1, f_2) \leq d_1 d_2.$$

Proof. See e.g. [Wal78, p. 59]. \square

Lemma 61. *Let s be a moment sequence for $\mathcal{B}_{3,2d}$. Suppose $p \in \mathcal{B}_{3,k}$ is irreducible in $\mathbb{R}[x_1, x_2, x_3]$, $k \leq d$, and $L_s(p^2(x_1^2 + x_2^2 + x_3^2)^{d-k}) = 0$. Then*

$$\mathcal{C}_{2d}(s) \leq dk + 1.$$

Proof. Consider the moment cone $\tilde{\mathcal{S}} := \mathcal{S}(\mathcal{B}_{3,2d}, \mathcal{Z}(p))$. Then $\tilde{\mathcal{S}}$ is an exposed face of the moment cone $\mathcal{S} = \mathcal{S}(\mathcal{B}_{3,2d}, \mathbb{P}(\mathbb{R}^2))$ and $s \in \tilde{\mathcal{S}}$. By Proposition 21, \mathcal{S} is closed and so is $\tilde{\mathcal{S}}$. Clearly, each point of $\tilde{\mathcal{S}}$ is the limit of relative inner points of $\tilde{\mathcal{S}}$. Therefore, since the sets $\tilde{\mathcal{S}}_k$ are closed by Proposition 21, it is sufficient to prove the assertion for all relatively inner points of the cone $\tilde{\mathcal{S}}$.

Let s be a relatively inner point of $\tilde{\mathcal{S}}$ and $x \in \mathcal{Z}(p)$. Setting $e := x_1^{2d} + x_2^{2d} + x_3^{2d}$, condition (6) holds. Since $\mathcal{Z}(p)$ is compact, Proposition 8 applies, so the supremum $c_s(x) := \sup \{c : s - c \cdot s_{\mathcal{B}_{3,2d}}(x) \in \tilde{\mathcal{S}}\}$ is attained and $s' := s - c_s(x) \cdot s_{\mathcal{B}_{3,2d}}(x) \in \partial \tilde{\mathcal{S}}$. Thus there exists a supporting hyperplane of the cone $\tilde{\mathcal{S}}$ at s' . Hence there exists a polynomial $q \in \mathcal{B}_{3,2d}$ such that $L_{s'}(q) = 0$, $L_s(q) > 0$, and $q \geq 0$ on $\mathcal{Z}(p)$. From $L_{s'}(q) = L_s(q) - c_s(x)q(x) = 0$ it follows that $q(x) \neq 0$. (Indeed, otherwise $L_s(q) = 0$, so s would be a boundary point of $\tilde{\mathcal{S}}$, a contradiction.) Since $p(x) = 0$ and $q(x) \neq 0$, the irreducible polynomial p is not a factor of q , so p and q are relatively prime and Bezout's Theorem applies.

Since $q(x) \geq 0$ on $\mathcal{Z}(p)$, for each intersection point of q and p has the intersection multiplicity of at least 2. Therefore, by Lemma 60,

$$(46) \quad 2|\mathcal{Z}(q) \cap \mathcal{Z}(p)| \leq \deg(q) \deg(p) = 2dk.$$

Since each representing measure of s' is supported on $\mathcal{Z}(p) \cap \mathcal{Z}(q)$, (46) implies that $\mathcal{C}_{2d}(s') \leq dk$. Hence $\mathcal{C}_{2d}(s) \leq \mathcal{C}_{2d}(s') + 1 \leq dk + 1$. \square

Our main result in this section is the following theorem.

Theorem 62. $\mathcal{C}_{2d} \leq \alpha(2d) + 1 = \frac{3}{2}d(d-1) + 2$ for $d \in \mathbb{N}$, $d \geq 5$.

Proof. Let us consider the moment cone $\mathcal{S} := \mathcal{S}(\mathcal{B}_{3,2d}, \mathbb{P}(\mathbb{R}^2))$. We proceed in a similar manner as in the proof of Lemma 61. By Proposition 21, the sets \mathcal{S}_k are closed. Hence it suffices to prove the inequality $\mathcal{C}_{2d}(s) \leq \alpha(2d) + 1$ for all relatively inner points of the cone \mathcal{S} .

Let s be an inner point of \mathcal{S} and $x \in \mathbb{P}(\mathbb{R}^2)$. By Proposition 8, the supremum $c_s(x) := \sup \{c : s - c \cdot s_{\mathcal{B}_{3,2d}}(x) \in \mathcal{S}\}$ is attained and $s' := s - c_s(x) \cdot s_{\mathcal{B}_{3,2d}}(x) \in \partial \mathcal{S}$. Then there exists a supporting hyperplane of \mathcal{S} at s , hence there is a polynomial

$f \in \mathcal{B}_{3,2d}$ such that $L_{s'}(f) = 0$ and $f \geq 0$ on $\mathbb{P}(\mathbb{R}^2)$. We apply Proposition 55 to f . Then, we can write $f = p \cdot q_1^2 \cdots q_r^2$ ($r \leq d$), where $p \in \text{Pos}(\mathbb{P}(\mathbb{R}^2))$, all q_i are indefinite and irreducible in $\mathbb{R}[x_1, x_2, x_3]$, $\mathcal{Z}(p) < \infty$ and all $|\mathcal{Z}(q_i)|$ are infinite. Since

$$\mathcal{Z}(f) = \mathcal{Z}(p) \cup \mathcal{Z}(q_1) \cup \cdots \cup \mathcal{Z}(q_r)$$

we find a disjoint decomposition $Z \cup Z_1 \cup \cdots \cup Z_r$ of $\mathcal{Z}(f)$ with $Z \subseteq \mathcal{Z}(p)$ and $Z_i \subseteq \mathcal{Z}(q_i)$. Let $\mu' = \sum_{j=1}^m c_j \delta_{x_j}$ be a representing measure of s' and set

$$s_0 := \sum_{x_j \in Z} c_j s_{\mathcal{B}_{3,2d}}(x_j) \quad \text{and} \quad s_i := \sum_{x_j \in Z_i} c_j s_{\mathcal{B}_{3,2d}}(x_j).$$

Clearly, $s' = s_0 + s_1 + \cdots + s_r$. Setting $d_i = \deg(q_i)$ and $2k = \deg(p)$, we have $d = k + d_1 + \cdots + d_r$ and $r \leq d - k$. Using Proposition 55 and Lemma 61 we derive

$$\begin{aligned} \mathcal{C}_{\mathcal{B}_{3,2d}}(s') &\leq \mathcal{C}_{\mathcal{B}_{3,2d}}(s_0) + \mathcal{C}_{\mathcal{B}_{3,2d}}(s_1) + \cdots + \mathcal{C}_{\mathcal{B}_{3,2d}}(s_r) \\ &\leq \alpha(2k) + (d \cdot d_1 + 1) + \cdots + (d \cdot d_r + 1) = \alpha(2k) + d(d - k) + r \\ &\leq \alpha(2k) + (d + 1)(d - k) = \alpha(2d) - \underbrace{(\alpha(2d) - \alpha(2k) - (d + 1)(d - k))}_{=\frac{1}{2}(d-k)(d+3k-5) \geq 0 \quad \forall d \geq 5, k=0, \dots, d} \\ &\leq \alpha(2d). \end{aligned}$$

Therefore, $\mathcal{C}_{\mathcal{B}_{3,2d}}(s) \leq \mathcal{C}_{\mathcal{B}_{3,2d}}(s') + 1 \leq \alpha(2d) + 1 = \frac{3}{2}d(d - 1) + 2$ for all $d \geq 5$. \square

Example 63 ($d = 5$). *W. R. Harris* [Har99] discovered a polynomial $h \in \mathcal{B}_{3,10}$ that is nonnegative on $\mathbb{P}(\mathbb{R}^2)$ with projective zero set

$$\mathcal{Z}_{\mathbb{P}}(h) = \{(1, 1, 0)^*, (1, 1, \sqrt{2})^*, (1, 1, 1/2)^*\},$$

where $(a, b, c)^*$ denotes all permutations of (a, b, c) including sign changes. Hence h has exactly 30 projective zeros $z_i, i = 1, \dots, 30$. A computer calculation shows that the matrix $(s_{\mathcal{B}_{3,10}}(z))_{z \in \mathcal{Z}_{\mathbb{P}}(h)}$ has rank 30, i.e., the set $\{s_{\mathcal{B}_{3,10}}(z_i) : i = 1, \dots, 30\}$ is linearly independent. Therefore, $\mathcal{C}_{\mathcal{B}_{3,10}} \geq 30$ by Theorem 18. Further, we compute $\mathcal{N}_{\mathcal{B}_{3,10}} = 15$ and have $\mathcal{C}_{\mathcal{B}_{3,10}} \leq \alpha(10) + 1 = 37$ by Theorem 62. Summarizing,

$$(47) \quad \mathcal{N}_{\mathcal{B}_{3,10}} = 15 < 30 \leq \mathcal{C}_{\mathcal{B}_{3,10}} \leq 37.$$

The following corollary reformulates Theorem 18 in the present context.

Corollary 64. *Let $d \in \mathbb{N}$ and $p \in \mathcal{B}_{3,2d}$. Suppose that $p \in \text{Pos}(\mathbb{R}^3)$, $|\mathcal{Z}(p)| = \beta(2d)$, and the set $\{s_{\mathcal{B}_{3,2d}}(z) : z \in \mathcal{Z}(p)\}$ is linearly independent. Then*

$$\beta(2d) \leq \mathcal{C}_{\mathcal{B}_{3,2d}}.$$

It seems natural to ask whether or not the assumption on the linear independence of the set $\{s_{\mathcal{B}_{3,2d}}(z) : z \in \mathcal{Z}(p)\}$ in Corollary 64 can be omitted. This leads to the

Question: *Suppose $p \in \mathcal{B}_{3,2d}$, $p \in \text{Pos}(\mathbb{R}^3)$, and $|\mathcal{Z}(p)| < \infty$ (or $|\mathcal{Z}(p)| = \beta(2d)$). Is the set $\{s_{\mathcal{B}_{3,2d}}(z) : z \in \mathcal{Z}(p)\}$ linearly independent?*

Note that for the Robinson polynomial $R \in \mathcal{B}_{3,6}$ the answer is “Yes”.

Recall that $\beta(2d) \leq \alpha(2d)$ by the Choi–Lam–Reznick Theorem (Proposition 55). It seems likely to conjecture that

$$(48) \quad \textbf{Conjecture : } \beta(2d) \leq \mathcal{C}_{\mathcal{B}_{3,2d}} \leq \beta(2d) + 1 \quad \text{for } d \geq 3.$$

The Robinson polynomial has 10 projective zeros, so that $\alpha(6) = \beta(6) = 10$. Therefore, since $\mathcal{C}_{\mathcal{B}_{3,6}} = 11$ as shown in [Kun14], this conjecture is true for $d = 3$. As noted above, the Harris polynomial $R \in \mathcal{B}_{3,10}$ has 30 projective zeros. Hence $30 \leq \beta(10) \leq \alpha(10) = 31$.

From the proof of Theorem 62 it follows that (48) holds if

$$\beta(d) + (d' + 1)(d' - d) \leq \beta(d') \quad \text{for } d' \in \mathbb{N}, d \in \mathbb{N}_0, d < d', (d', d) \neq (3, 0).$$

7. CARATHÉODORY NUMBERS AND REAL WARING RANK

In Definition 4 we introduced the signed Carathéodory number $\mathcal{C}_{A,\pm}$. In this section we connect it to the *real Waring rank* $w(n, 2d)$, that is, to the smallest number $w(n, 2d)$ such that each $f \in \mathcal{B}_{n,2d}$ can be written as real linear combination

$$(49) \quad f(x) = \sum_{i=1}^k c_i (x \cdot \lambda_i)^{2d}$$

of $2d$ -powers of linear forms $x \cdot \lambda_i = \lambda_{i,1}x_1 + \dots + \lambda_{i,n}x_n$, where $k \leq w(n, 2d)$, $c_i \in \mathbb{R}$, $\lambda_i \in \mathbb{R}^n$.

Let us recall some basics on the *apolar scalar product* $[\cdot, \cdot]$, see e.g. [Rez92]. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| := \alpha_1 + \dots + \alpha_n = 2d$ we set $\gamma_\alpha := \frac{(2d)!}{\alpha_1! \dots \alpha_n!}$. Let $p, q \in \mathcal{B}_{n,2d}$. We write $p(x) = \sum_\alpha \gamma_\alpha a_\alpha x^\alpha$ and $q(x) = \sum_\alpha \gamma_\alpha b_\alpha x^\alpha$ and define

$$[p, q] := \sum_\alpha \gamma_\alpha a_\alpha b_\alpha.$$

Then $(\mathcal{B}_{n,2d}, [\cdot, \cdot])$ becomes a finite-dimensional real Hilbert space. Setting $f_\lambda(x) := (\lambda \cdot x)^{2d}$, we obtain

$$(50) \quad [p, f_\lambda] = \sum_\alpha \gamma_\alpha a_\alpha \lambda^\alpha = p(\lambda).$$

Let f be of the form (49). Then, for $p \in \mathcal{B}_{n,2d}$ it follows from (50) that

$$(51) \quad L_f(p) := [f, p] = \left[\sum_{i=1}^k c_i f_{\lambda_i}, p \right] = \sum_{i=1}^k c_i p(\lambda_i),$$

that is, the linear functional L_f on $\mathcal{B}_{n,2d}$ is the integral with respect to the signed measure $\mu := \sum_{i=1}^k c_i \delta_{\lambda_i}$. Conversely, each signed atomic measure yields a function f of the form (49) such that (51) holds. By the Riesz Theorem all linear functionals on $\mathcal{B}_{n,2d}$ are of the form L_f , where f is as in (49).

Theorem 65. (i) $w(n, 2d) = \mathcal{C}_{\mathcal{B}_{n,2d}, \pm}$.

(ii) $\mathcal{N}_{\mathcal{B}_{n,2d}} \leq w(n, 2d) \leq 2\mathcal{N}_{\mathcal{B}_{n,2d}}$.

(iii) Set $N := \mathcal{N}_{\mathcal{B}_{n,2d}}$. Then there exists $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^{N \cdot n}$ such that for all $\varepsilon > 0$ and $p \in \mathcal{B}_{n,2d}$ we have

$$p(x) = c \cdot \sum_{i=1}^{\mathcal{N}_{\mathcal{B}_{n,2d}}} [(\lambda_i \cdot x)^{2d} - c_i (\lambda_i^\varepsilon \cdot x)^{2d}]$$

for some $\lambda^\varepsilon = (\lambda_1^\varepsilon, \dots, \lambda_N^\varepsilon)$ with $\|\lambda - \lambda^\varepsilon\| < \varepsilon$, $|1 - c_i| < \varepsilon$, $c \in \mathbb{R}$.

(iv) The set of vectors λ as in (iii) is open and dense in $\mathbb{R}^{\mathcal{N}_{\mathcal{B}_{n,2d}} \cdot n}$.

Proof. (i) is clear from the preceding considerations on the apolar scalar product. Remark 28 and (i) imply (ii), while (iii) follows from Theorem 25 combined with (i). (iv) is a consequence of Sard's Theorem as in Theorem 27. \square

With Theorem 53 the upper bound in (ii) was already obtained in [BT15, Cor. 9].

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